

Fermion masses and neutrino mixing in an $U(1)_H$ flavor symmetry model with hierarchical radiative generation for light charged fermion masses

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Abstract

I report the analysis performed on fermion masses and mixing, including neutrino mixing, within the context of a model with hierarchical radiative mass generation mechanism for light charged fermions, mediated by exotic scalar particles at one and two loops, respectively, meanwhile the neutrinos get Majorana mass terms at tree level through the Yukawa couplings with two $SU(2)_L$ Higgs triplets. All the resulting mass matrices in the model, for the u, d, and e fermion charged sectors, the neutrinos and the exotic scalar particles, are diagonalized in exact analytical form. Quantitative analysis shows that this model is successful to accommodate the hierarchical spectrum of masses and mixing in the quark sector as well as the charged lepton masses. The lepton mixing matrix, V_{PMNS} , is written completely in terms of the neutrino masses m_1, m_2 , and m_3 . Large lepton mixing for θ_{12} and θ_{23} is predicted in the range of values $0.7 \lesssim \sin^2 2\theta_{12} \lesssim 0.7772$ and $0.87 \lesssim \sin^2 2\theta_{23} \lesssim 0.9023$ by using $0.033 \lesssim s_{13}^2 \lesssim 0.04$. These values for lepton mixing are consistent with 3σ allowed ranges provided by recent global analysis of neutrino data oscillation. From Δm_{sol}^2 bounds, neutrino masses are predicted in the range of values $m_1 \approx (1.706 - 2.494) \times 10^{-3} \text{eV}$, $m_2 \approx (6.675 - 12.56) \times 10^{-3} \text{eV}$, and $m_3 \approx (1.215 - 2.188) \times 10^{-2} \text{eV}$, respectively. The above allowed lepton mixing leads to the quark-lepton complementary relations $\theta_{12}^{CKM} + \theta_{12}^{PMNS} \approx 41.543^\circ - 44.066^\circ$ and $\theta_{23}^{CKM} + \theta_{23}^{PMNS} \approx 36.835^\circ - 38.295^\circ$. The new exotic scalar particles induce flavor changing neutral currents and contribute to lepton flavor violating processes such as $E \rightarrow e_1 e_2 e_3$, to radiative rare decays, $\tau \rightarrow \mu \gamma, \tau \rightarrow e \gamma, \mu \rightarrow e \gamma$, as well as to the anomalous magnetic moments of fermions. I give general analytical expressions for the branching ratios of these rare decays and for the anomalous magnetic moments for charged leptons.

Keywords: Neutrino mixing, Fermion masses and mixing, Flavor symmetry.

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1 Introduction

The observed hierarchical spectrum of masses and mixing angles in the quark sector still remains as one of the most important challenges in particle physics. A possible solution to explain this hierarchical spectrum is that light fermion masses arise through radiative corrections [1], while the masses for top quark, bottom quark, and tau lepton are generated either at the tree level like in Ref.[2], or by the implementation of seesaw-type mechanisms as was proposed by the author in a model with a $SU(3)$ horizontal symmetry in Ref. [3]. Recently, it has also been possible to observe the phenomenon of flavor mixing in the leptonic sector

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through the confirmation of the phenomenon of "neutrino oscillation" in experiments of neutrinos coming from atmospheric [4], solar [5], reactors [6] and accelerators [7], and as a consequence the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) lepton mixing matrix, V_{PMNS} , has been determined. For three flavor neutrinos ν_e , ν_μ , and ν_τ , which may evolve into the three massive neutrinos ν_1 , ν_2 , ν_3 , the experiments of neutrino oscillations are interpreted in terms of three mixing angles denoted by θ_{12} for $\nu_e - \nu_\mu$, θ_{23} for $\nu_\mu - \nu_\tau$, and θ_{13} for $\nu_e - \nu_\tau$. Recent analysis and fit of neutrino oscillation data give [8], at the 3σ level, the allowed ranges of values

$$\begin{aligned} |\Delta m_{23}^2| &= (1.4 - 3.3) \times 10^{-3} (eV)^2, & \Delta m_{12}^2 &= (7.1 - 8.9) \times 10^{-5} (eV)^2, \\ \sin^2 2\theta_{23} &= 0.87 - 1.0, & \sin^2 2\theta_{12} &= 0.70 - 0.94, \\ \sin^2 \theta_{13} &\leq 0.051, \end{aligned} \quad (1)$$

where Δm_{12}^2 and Δm_{23}^2 are the solar and atmospheric mass differences, respectively.

In this article I address the problem of fermion masses and mixing angles, including neutrino mixing, within the context of the model introduced in Ref.[2]. Section 2 briefly reviews the main features of the model with an $U(1)_H$ flavor symmetry. Next, in Sec. 3 I discuss the masses and mixing for charged leptons and quarks, at one and two loops, and using the strong hierarchy of masses, approximate mixing matrices for the u, d, and e charged sectors are provided. Section 4 is devoted to finding the upper bounds for mixing angles of charged leptons. In Sec. 5 I analyze neutrino mixing, and the V_{PMNS} lepton mixing matrix is written in terms of neutrino masses, providing numerical results for neutrino mixing. In Sec. 6 I perform a quantitative analysis of quarks masses and mixing, including numerical values for the V_{CKM} . In Sec. 7 I give general expressions for the flavor changing neutral currents (FCNCs), rare decays and anomalous magnetic moments for charged leptons that are induced by the exotic scalar particles. Section 8 contains my conclusions. In the Appendix I have introduced a method to diagonalize in close analytical form a generic 3x3 real and symmetric mass matrix, and then I have extended this method to diagonalize the 4x4 real symmetric exotic scalar mass matrix.

2 Model with $U(1)_H$ flavor symmetry

The gauge symmetry of the model is defined as $U(1)_H \otimes SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$. The fermionic content of the model is the same as in the "standard model" (SM), and their charges under the flavor symmetry $U(1)_H$ are arranged as to cancel anomalies without the introduction of exotic fermions. The fermions are classified, as in the SM, in five sectors $f = q, u, d, l$, and e , where q and l are the $SU(2)_L$ quark and lepton doublets, respectively, and u, d , and e are the singlets, in an obvious notation.

The cancelation of anomalies in a simple way that simultaneously guarantees that only the third generation of the charged fermions acquire masses at tree level is given by [2]

$$H(f) = 0, \pm\delta_f \quad , \quad \delta_q^2 - 2\delta_u^2 + \delta_d^2 = \delta_l^2 - \delta_e^2, \quad (2)$$

with the constraints

$$\delta_l = \delta_q = \Delta \neq \delta_u = \delta_d = \delta_e = \delta \quad (3)$$

The assignment of flavor charges to the fermions is then as given in Table 1. The $G_{SM} \equiv SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ quantum numbers of fermions are the same as in the SM.

The particle content of the model is such that we can implement a hierarchical mass generation mechanism, where the third family of charged fermions obtain mass at tree level, while the light charged fermions

Sector	Family 1	Family 2	Family 3
q	Δ	$-\Delta$	0
u	δ	$-\delta$	0
d	δ	$-\delta$	0
l	Δ	$-\Delta$	0
e	δ	$-\delta$	0

Table 1: Assignment of family charges under $U(1)_H$.

quantum number	Class I		Class II						Class I		Class II	
	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9	ϕ_{10}	ϕ_{11}	ϕ_{12}
H	0	$-\delta$	0	Δ	0	δ	0	δ	Δ	0	δ	0
Y	1	0	$-\frac{2}{3}$	$-\frac{2}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$-\frac{8}{3}$	$-\frac{8}{3}$	2	2	4	4
T	$\frac{1}{2}$	0	1	1	0	0	0	0	1	1	0	0
C	1	1	$\bar{6}$	$\bar{6}$	$\bar{6}$	$\bar{6}$	$\bar{6}$	$\bar{6}$	1	1	1	1

Table 2: Assignment of charges for scalar fields under $U(1)_H \otimes SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$.

get masses at one and two loops, respectively. In particular, the radiative mass generation for the light charged leptons involve the introduction of two $SU(2)_L$ weak scalar triplets with neutral fields, which we allow to get vacuum expectation values (VEVs). The VEVs of these triplets contribute very little to the W and Z masses and simultaneously allow the generation of tree level Majorana mass terms for the left-handed neutrinos.

The scalar fields introduced in the model are then divided into two classes. Class I (II) contains scalar fields which acquire (do not acquire) VEV. These scalar fields are as given in Table 2.

The Yukawa couplings are classified in two types, Dirac(D) and Majorana(M)[Figs. (1a) and (1b)], respectively,

$$\mathcal{L}_Y = \mathcal{L}_{YD} + \mathcal{L}_{YM}, \quad (4)$$

where

$$\mathcal{L}_{YD} = Y^u \bar{q}_{L3} \tilde{\phi}_1 u_{R3} + Y^d \bar{q}_{L3} \phi_1 d_{R3} + Y^e \bar{l}_{L3} \phi_1 \tau_{R3} + H.c., \quad (5)$$

with $\bar{\Psi} = \Psi^\dagger \gamma^0$, $\tilde{\phi} = i\sigma_2 \phi^*$, Y^i , where $i = u, d, e$ are coupling constants, and

$$\begin{aligned} \mathcal{L}_{YM} = & Y^q_{12} q_{1L}^\alpha{}^T C \phi_{3\{\alpha,\beta\}} q_{2L}^\beta + Y^q_{23} q_{2L}^\alpha{}^T C \phi_{4\{\alpha,\beta\}} q_{3L}^\beta + Y^q_{33} q_{3L}^\alpha{}^T C \phi_{3\{\alpha,\beta\}} q_{3L}^\beta \\ & + Y^u_{12} u_{1R}{}^T C \phi_7 u_{2R} + Y^u_{23} u_{2R}{}^T C \phi_8 u_{3R} + Y^u_{33} u_{3R}{}^T C \phi_7 u_{3R} \\ & + Y^d_{12} d_{1R}{}^T C \phi_5 d_{2R} + Y^d_{23} d_{2R}{}^T C \phi_6 d_{3R} + Y^d_{33} d_{3R}{}^T C \phi_5 d_{3R} + h.c. \end{aligned} \quad (6)$$

$$\begin{aligned} & + Y_{12} l_{1L}^\alpha{}^T C \phi_{10\{\alpha,\beta\}} l_{2L}^\beta + Y_{23} l_{2L}^\alpha{}^T C \phi_{9\{\alpha,\beta\}} l_{3L}^\beta + Y_{33} l_{3L}^\alpha{}^T C \phi_{10\{\alpha,\beta\}} l_{3L}^\beta \\ & + Y_{12} e_R{}^T C \phi_{12} \mu_R + Y_{23} \mu_R{}^T C \phi_{11} \tau_R + Y_{33} \tau_R{}^T C \phi_{12} \tau_R + h.c. \end{aligned} \quad (7)$$

In these couplings C represents the charge conjugation matrix, α and β are weak isospin indices, and color indices have been omitted. Couplings of Eq.(6) are introduced for the quark sector, while those of Eq.(7) are needed for the lepton sector.

Notice that ϕ_3 and ϕ_9 are represented as

$$\phi_3 = \begin{pmatrix} \phi^{-4/3} & \phi^{-1/3} \\ \phi^{-1/3} & \phi^{2/3} \end{pmatrix}, \text{ and } \phi_9 = \begin{pmatrix} \phi^0 & \phi^+ \\ \phi^+ & \phi^{++} \end{pmatrix}, \quad (8)$$

where the superscripts denote the electric charge of the fields (and corresponding expressions for ϕ_4 and ϕ_{10}).

The most general scalar potential is written as

$$\begin{aligned} -V(\phi_i) = & \sum_i \mu_i^2 \|\phi_i\|^2 + \sum_{i,j} \lambda_{ij} \|\phi_i\|^2 \|\phi_j\|^2 + \eta_{31} \phi_1^\dagger \phi_3^\dagger \phi_3 \phi_1 + \tilde{\eta}_{31} \tilde{\phi}_1^\dagger \phi_3^\dagger \phi_3 \tilde{\phi}_1 \\ & + \eta_{41} \phi_1^\dagger \phi_4^\dagger \phi_4 \phi_1 + \tilde{\eta}_{41} \tilde{\phi}_1^\dagger \phi_4^\dagger \phi_4 \tilde{\phi}_1 + \kappa_{91} \phi_1^\dagger \phi_9^\dagger \phi_9 \phi_1 + \tilde{\kappa}_{91} \tilde{\phi}_1^\dagger \phi_9^\dagger \phi_9 \tilde{\phi}_1 \\ & + \kappa_{10,1} \phi_1^\dagger \phi_{10}^\dagger \phi_{10} \phi_1 + \tilde{\kappa}_{10,1} \tilde{\phi}_1^\dagger \phi_{10}^\dagger \phi_{10} \tilde{\phi}_1 + \sum_{i \neq j, j \neq 1,2} \eta_{ij} \|\phi_i^\dagger \phi_j\|^2 \\ & + (\rho_1 \phi_5^\dagger \phi_6 \phi_2 + \rho_2 \phi_7^\dagger \phi_8 \phi_2 + \lambda_1 \phi_5^\dagger \phi_1^\alpha \phi_{3\{\alpha,\beta\}} \phi_1^\beta + \lambda_2 \phi_7^\dagger \phi_1^\alpha \phi_{3\{\alpha,\beta\}} \phi_1^\beta \\ & + \lambda_3 Tr(\phi_3^\dagger \phi_4) \phi_2^2 + \lambda_4 \phi_5 \phi_6 \phi_7 \phi_2 + \lambda_5 \phi_5 \phi_6^\dagger \phi_7^\dagger \phi_8 + \lambda_6 \phi_2 \phi_8 \phi_5^2 + y_1 \phi_{12}^\dagger \phi_{11} \phi_2 \\ & + \zeta_1 \phi_{12}^2 \phi_1^\alpha \phi_{10\{\alpha,\beta\}} \phi_1^\beta + Y_r Tr(\phi_{10}^\dagger \phi_9) \phi_2^2 + \epsilon_1 \phi_5 \phi_6^\dagger \phi_{12}^\dagger \phi_{11} \\ & + \epsilon_2 \phi_7^\dagger \phi_8 \phi_{12} \phi_{11}^\dagger + h.c.) \end{aligned} \quad (9)$$

where Tr means trace and in $\|\phi_i\|^2 = \phi_i^\dagger \phi_i$ an appropriate contraction of the $SU(2)_L$ and $SU(3)_C$ indices is understood. The gauge invariance of this potential requires the relation $\Delta = 2\delta$ to be hold.

In order to break the symmetry, the VEVs for the class I scalar fields are assumed to be in the form

$$\langle \phi_1 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \quad \langle \phi_2 \rangle = v_2, \quad (10)$$

$$\langle \phi_9 \rangle = \begin{pmatrix} v_9 & 0 \\ 0 & 0 \end{pmatrix}, \quad \langle \phi_{10} \rangle = \begin{pmatrix} v_{10} & 0 \\ 0 & 0 \end{pmatrix}. \quad (11)$$

$\langle \phi_1 \rangle$ and $\langle \phi_2 \rangle$ achieve the symmetry breaking sequence

$$U(1)_H \otimes G_{SM} \xrightarrow{\langle \phi_2 \rangle} G_{SM} \xrightarrow{\langle \phi_1 \rangle} SU(3)_C \otimes U(1)_Q, \quad (12)$$

while the VEVs v_9 and v_{10} are extremely small in order to be consistent with the experimental bounds on the ρ parameter. $M_W = \frac{1}{2} g v_1$ with $v_1 \approx 246 \text{ GeV}$, and I assume v_2 in the TeV region.

The scalar field mixing arises after spontaneous symmetry breaking (SSB) from the terms in the potential that couple two different class II fields to one class I field. After SSB the mass matrix for the scalar fields of charge $+2$, $(\phi_9, \phi_{10}, \phi_{12}, \phi_{11})$ may be written as

$$M_{+2}^2 = \begin{pmatrix} e_9^2 & Y_r^* v_2^2 & 0 & 0 \\ Y_r v_2^2 & e_{10}^2 & \frac{\zeta_1^* v_1^2}{2} & 0 \\ 0 & \frac{\zeta_1 v_1^2}{2} & e_{12}^2 & y_1 v_2 \\ 0 & 0 & y_1^* v_2 & e_{11}^2 \end{pmatrix}, \quad (13)$$

where $e_i^2 = \mu_i^2 + \lambda_{i1} \frac{v_1^2}{2} + \lambda_{i2} v_2^2$ for $i = 9, 10, 11, 12$, and analogous ones for the $-\frac{4}{3}$ and $\frac{2}{3}$ scalar sectors.

3 Masses and mixing for charged fermions

Now I give a brief description of the hierarchical mass generation mechanism for the charged fermions. After the SSB of the electroweak symmetry down to $U(1)_Q$ of QED, the Yukawa couplings of Eq.(5) generate tree level masses for the top and bottom quarks and the τ lepton. For the light charged fermions, the scalar fields introduced in the model allow the one and two loop diagrams of Fig. 2 for the charged lepton mass matrix elements, and similar ones for the up and down quark sectors. In the diagrams of Fig. 2 the cross in the internal fermion lines means tree level mixing and the black dot means one loop mixing. The diagrams of Figs. 3(a) and 3(b) should be added to the matrix elements (1,3) and (3,1), respectively.

In the one loop contribution to the mass matrices for the different charged fermion sectors only the third family of fermions appears in the internal lines. This generate a rank 2 matrix, which once diagonalized gives the mass eigenstates at this approximation. Then using these mass eigenstates the next order contribution is computed, obtaining a matrix of rank 3. After the diagonalization of this last matrix the mass eigenvalues and eigenstates are obtained.

3.1 Charged leptons

The nonvanishing contributions from the one loop diagrams of Fig. 2 to the mass terms $\bar{e}_{iR}e_{jL}\Sigma_{ij}^{(1)}$ are

$$\Sigma_{22}^{(1)} = m_\tau^0 \frac{Y_{23}^2}{16\pi^2} \sum_k U_{1k} U_{4k} f(M_k, m_\tau^0), \quad (14)$$

$$\Sigma_{23}^{(1)} = m_\tau^0 \frac{Y_{23} Y_{33}}{16\pi^2} \sum_k U_{2k} U_{4k} f(M_k, m_\tau^0), \quad (15)$$

$$\Sigma_{32}^{(1)} = m_\tau^0 \frac{Y_{23} Y_{33}}{16\pi^2} \sum_k U_{1k} U_{3k} f(M_k, m_\tau^0), \quad (16)$$

where m_τ^0 is the tree level contribution, U is the orthogonal matrix which diagonalizes M_ϕ^2 , with

$$\Phi_i = U_{ij} \sigma_j \quad , \quad \Phi_1 \equiv \phi_9 \quad , \quad \Phi_2 \equiv \phi_{10} \quad , \quad \Phi_3 \equiv \phi_{12} \quad , \quad \Phi_4 \equiv \phi_{11} \quad (17)$$

being the relation between gauge and mass scalar eigenfields σ_i , $i, j = 1, 2, 3, 4$, M_k^2 are the scalar mass eigenvalues, and

$$f(a, b) \equiv \frac{a^2}{a^2 - b^2} \ln \frac{a^2}{b^2} . \quad (18)$$

Thus, the one loop contribution to fermion masses may be written as

$$M_e^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Sigma_{22}^{(1)} & \Sigma_{23}^{(1)} \\ 0 & \Sigma_{32}^{(1)} & m_\tau^0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_2 & a_{23} \\ 0 & a_{32} & a_3 \end{pmatrix} . \quad (19)$$

Now, $M_e^{(1)}$ is diagonalized by a biunitary transformation

$$V_R^{(1)\dagger} M_e^{(1)} V_L^{(1)} = M_D^{(1)} , \quad (20)$$

$$M_e^{(1)T} M_e^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (a_2^2 + a_{32}^2) & (a_2 a_{23} + a_3 a_{32}) \\ 0 & (a_2 a_{23} + a_3 a_{32}) & (a_3^2 + a_{23}^2) \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & a'_L & c'_L \\ 0 & c'_L & b'_L \end{pmatrix}, \quad (21)$$

$$M_e^{(1)} M_e^{(1)T} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (a_2^2 + a_{23}^2) & (a_2 a_{32} + a_3 a_{23}) \\ 0 & (a_2 a_{32} + a_3 a_{23}) & (a_3^2 + a_{32}^2) \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & a'_R & c'_R \\ 0 & c'_R & b'_R \end{pmatrix}, \quad (22)$$

where in this report (ignoring CP violation), the orthogonal matrices $V_L^{(1)}$ and $V_R^{(1)}$ are given by ¹

$$V_L^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha_L & -\sin \alpha_L \\ 0 & \sin \alpha_L & \cos \alpha_L \end{pmatrix}, \quad V_R^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha_R & -\sin \alpha_R \\ 0 & \sin \alpha_R & \cos \alpha_R \end{pmatrix}, \quad (23)$$

where

$$\begin{aligned} \cos \alpha_L &= \alpha'(\lambda_+ - a'_L) = \sqrt{\frac{\lambda_+ - a'_L}{\lambda_+ - \lambda_-}}, & \cos \alpha_R &= \beta'(\lambda_+ - a'_R) = \sqrt{\frac{\lambda_+ - a'_R}{\lambda_+ - \lambda_-}}, \\ \sin \alpha_L &= -\alpha'c'_L = \sqrt{\frac{\lambda_+ - b'_L}{\lambda_+ - \lambda_-}}, & \sin \alpha_R &= -\beta'c'_R = \sqrt{\frac{\lambda_+ - b'_R}{\lambda_+ - \lambda_-}}, \end{aligned} \quad (24)$$

λ_- and λ_+ are the solutions of the equation

$$\lambda^2 - B'\lambda + D' = 0, \quad (25)$$

$$\lambda_- = \frac{1}{2} [B' - \sqrt{B'^2 - 4D'}], \quad \lambda_+ = \frac{1}{2} [B' + \sqrt{B'^2 - 4D'}], \quad (26)$$

$$\begin{aligned} B' &= a'_L + b'_L = a'_R + b'_R = a_2^2 + a_3^2 + a_{23}^2 + a_{32}^2 = \lambda_- + \lambda_+, \\ D' &= a'_L b'_L - c'^2_L = a'_R b'_R - c'^2_R = (a_2 a_3 - a_{23} a_{32})^2 = \lambda_- \lambda_+, \end{aligned} \quad (27)$$

$$a_2 a_3 - a_{23} a_{32} > 0,$$

and

$$\alpha' \equiv \frac{1}{\sqrt{c'^2_L + (\lambda_+ - a'_L)^2}}, \quad \beta' \equiv \frac{1}{\sqrt{c'^2_R + (\lambda_+ - a'_R)^2}}, \quad (28)$$

$$V_L^{(1)T} M_e^{(1)T} M_e^{(1)} V_L^{(1)} = V_R^{(1)T} M_e^{(1)} M_e^{(1)T} V_R^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_- & 0 \\ 0 & 0 & \lambda_+ \end{pmatrix}, \quad (29)$$

$$V_R^{(1)T} M_e^{(1)} V_L^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{\lambda_-} & 0 \\ 0 & 0 & \sqrt{\lambda_+} \end{pmatrix}. \quad (30)$$

¹I assume the signs: $a_2 > 0$, $a_{23} < 0$ and $a_{32} < 0$

Therefore, from Eq. (30), up to one loop level

$$m_1^{(1)} = 0 \quad , \quad m_2^{(1)} = \sqrt{\lambda_-} \quad , \quad m_3^{(1)} = \sqrt{\lambda_+} \quad , \quad (31)$$

with the expected hierarchy $\lambda_- \ll \lambda_+$.

Two loop contributions for charged leptons:

$$\Sigma_{11}^{(2)} = \frac{Y_{12}^2}{16\pi^2} \sum_{k,i} m_i^{(1)} (V_L^{(1)})_{2i} (V_R^{(1)})_{2i} U_{2k} U_{3k} f(M_k, m_i^{(1)}), \quad (32)$$

$$\Sigma_{12}^{(2)} = \frac{Y_{12} Y_{23}}{16\pi^2} \sum_{k,i} m_i^{(1)} (V_L^{(1)})_{3i} (V_R^{(1)})_{2i} U_{1k} U_{3k} f(M_k, m_i^{(1)}), \quad (33)$$

$$\begin{aligned} \Sigma_{13}^{(2)} &= \frac{Y_{12} Y_{23}}{16\pi^2} \sum_{k,i} m_i^{(1)} (V_L^{(1)})_{2i} (V_R^{(1)})_{2i} U_{1k} U_{3k} f(M_k, m_i^{(1)}) \\ &\quad + \frac{Y_{12} Y_{33}}{16\pi^2} \sum_{k,i} m_i^{(1)} (V_L^{(1)})_{3i} (V_R^{(1)})_{2i} U_{2k} U_{3k} f(M_k, m_i^{(1)}) \quad , \end{aligned} \quad (34)$$

$$\Sigma_{21}^{(2)} = \frac{Y_{12} Y_{23}}{16\pi^2} \sum_{k,i} m_i^{(1)} (V_L^{(1)})_{2i} (V_R^{(1)})_{3i} U_{2k} U_{4k} f(M_k, m_i^{(1)}), \quad (35)$$

$$\begin{aligned} \Sigma_{31}^{(2)} &= \frac{Y_{12} Y_{23}}{16\pi^2} \sum_{k,i} m_i^{(1)} (V_L^{(1)})_{2i} (V_R^{(1)})_{2i} U_{2k} U_{4k} f(M_k, m_i^{(1)}) \\ &\quad + \frac{Y_{12} Y_{33}}{16\pi^2} \sum_{k,i} m_i^{(1)} (V_L^{(1)})_{2i} (V_R^{(1)})_{3i} U_{2k} U_{3k} f(M_k, m_i^{(1)}) \quad , \end{aligned} \quad (36)$$

where $i = 2, 3$ and $k = 1, 2, 3, 4$.

Hence the mass matrix for charged leptons up to two loop contributions may be written in good approximation as

$$M_e^{(2)} \approx \begin{pmatrix} \Sigma_{11}^{(2)} & \Sigma_{12}^{(2)} & \Sigma_{13}^{(2)} \\ \Sigma_{21}^{(2)} & \sqrt{\lambda_-} & 0 \\ \Sigma_{31}^{(2)} & 0 & \sqrt{\lambda_+} \end{pmatrix} \equiv \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \sqrt{\lambda_-} & 0 \\ \Sigma_{31} & 0 & \sqrt{\lambda_+} \end{pmatrix}. \quad (37)$$

In the limit $M_k \gg m_\tau^0$ the function $f(a, b)$ behaves as $\ln \frac{a^2}{b^2}$. In this limit, and introducing the $m_i^{(1)}$ one loop mass eigenvalues, Eq.(31), the orthogonal matrices $V_L^{(1)}$ and $V_R^{(1)}$, Eq.(23), the relationships

$$\begin{aligned} \cos \alpha_L \cos \alpha_R &= \frac{a_3 \sqrt{\lambda_+} - a_2 \sqrt{\lambda_-}}{\lambda_+ - \lambda_-} \quad , \quad \cos \alpha_L \sin \alpha_R = - \frac{a_{23} \sqrt{\lambda_+} + a_{32} \sqrt{\lambda_-}}{\lambda_+ - \lambda_-} \quad , \\ \sin \alpha_L \sin \alpha_R &= \frac{a_2 \sqrt{\lambda_+} - a_3 \sqrt{\lambda_-}}{\lambda_+ - \lambda_-} \quad , \quad \sin \alpha_L \cos \alpha_R = - \frac{a_{32} \sqrt{\lambda_+} + a_{23} \sqrt{\lambda_-}}{\lambda_+ - \lambda_-} \quad , \end{aligned} \quad (38)$$

and using the orthogonality of U , one obtains explicitly

$$\begin{aligned}
\Sigma_{11} &= a_2 \sigma > 0 \quad , \\
\Sigma_{12} &= \Sigma_{21} = \frac{1}{c_\alpha} \frac{a_{23} a_{32}}{a_3} > 0 \quad , \quad c_\alpha \equiv \frac{Y_{33}}{Y_{12}} \\
\Sigma_{13} &= c_\alpha a_{23} \sigma + \frac{1}{c_\alpha} \frac{a_2 a_{32}}{a_3} < 0 \quad , \\
\Sigma_{31} &= c_\alpha a_{32} \sigma + \frac{1}{c_\alpha} \frac{a_2 a_{23}}{a_3} < 0 \quad ,
\end{aligned} \tag{39}$$

where the parameter σ is defined as

$$\sigma \equiv \frac{Y_{12}^2}{16\pi^2} \sum_k U_{2k} U_{3k} \ln \frac{M_k^2}{m_\tau^0} > 0 . \tag{40}$$

3.2 Quarks

Performing an analogous analysis, the one loop contributions for the down quark sector are

$$\left(\Sigma_{22}^{(1)} \right)^d = m_b^0 \frac{Y_{23}^q Y_{23}^d}{16\pi^2} \sum_k U_{1k}^d U_{4k}^d f(M_k^d, m_b^0) \equiv a_2^d , \tag{41}$$

$$\left(\Sigma_{23}^{(1)} \right)^d = m_b^0 \frac{Y_{33}^q Y_{23}^d}{16\pi^2} \sum_k U_{2k}^d U_{4k}^d f(M_k^d, m_b^0) \equiv a_{23}^d , \tag{42}$$

$$\left(\Sigma_{32}^{(1)} \right)^d = m_b^0 \frac{Y_{23}^q Y_{33}^d}{16\pi^2} \sum_k U_{1k}^d U_{3k}^d f(M_k^d, m_b^0) \equiv a_{32}^d , \tag{43}$$

where m_b^0 is the tree level contribution, U^d is the orthogonal matrix which diagonalizes $M_{d\phi}^2$, with

$$\Phi_i^d = U_{ij}^d \sigma_j^d \quad , \quad \Phi_1^d \equiv \phi_4 \quad , \quad \Phi_2^d \equiv \phi_3 \quad , \quad \Phi_3^d \equiv \phi_5 \quad , \quad \Phi_4^d \equiv \phi_6 \tag{44}$$

being the relation between gauge and mass scalar eigenfields σ_i^d , $i, j = 1, 2, 3, 4$ and M_k^{d2} are the mass eigenvalues.

Similarly, the two loop contributions for the down quark sector may be expressed as

$$\begin{aligned}
\Sigma_{11}^d &= a_2^d \sigma^d > 0, \\
\Sigma_{12}^d &= \frac{Y_{12}^d}{Y_{33}^d} \frac{a_{23}^d a_{32}^d}{m_b^0} > 0, \\
\Sigma_{21}^d &= \frac{Y_{12}^q}{Y_{33}^q} \frac{a_{23}^d a_{32}^d}{m_b^0} > 0, \\
\Sigma_{13}^d &= \frac{Y_{33}^q}{Y_{12}^q} a_{23}^d \sigma^d + \frac{Y_{12}^d}{Y_{33}^d} \frac{a_2^d a_{32}^d}{m_b^0} < 0, \\
\Sigma_{31}^d &= \frac{Y_{33}^d}{Y_{12}^d} a_{32}^d \sigma^d + \frac{Y_{12}^q}{Y_{33}^q} \frac{a_2^d a_{23}^d}{m_b^0} < 0,
\end{aligned} \tag{45}$$

with σ^d defined as

$$\sigma^d = \frac{Y_{12}^q Y_{12}^d}{16\pi^2} \sum_k U_{2k}^d U_{3k}^d \ln \frac{M_k^{d^2}}{m_b^0} > 0, \tag{46}$$

and analogous ones for the up quark sector.

Notice that the radiative corrections give rise to the following hierarchy among the parameters in the mass matrices for charged leptons

$$\Sigma_{11} \quad , \quad \Sigma_{12} \quad , \quad \Sigma_{21} \quad , \quad |\Sigma_{13}| \quad , \quad |\Sigma_{31}| \quad \ll \quad a_2 \quad , \quad |a_{23}| \quad , \quad |a_{32}| \quad \ll \quad a_3 = m_\tau^o, \tag{47}$$

and similarly for the u and d quark sectors.

The matrix $M_e^{(2)}$, Eq. (37), is diagonalized by a new biunitary transformation

$$V_R^{(2)\dagger} M_e^{(2)} V_L^{(2)} = M_D^{(2)}. \tag{48}$$

In this case, with the aid of results of the Appendix, the orthogonal matrix $V_L^{(2)}$ may be written as

$$V_L^{(2)} = \begin{pmatrix} \sqrt{\frac{(b_L - \lambda_1)(c_L - \lambda_1) - f_L^2}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)}} & \sqrt{\frac{(b_L - \lambda_2)(c_L - \lambda_2) - f_L^2}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)}} & -\sqrt{\frac{(b_L - \lambda_3)(c_L - \lambda_3) - f_L^2}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}} \\ -\sqrt{\frac{(a_L - \lambda_1)(c_L - \lambda_1) - e_L^2}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)}} & \sqrt{\frac{(a_L - \lambda_2)(c_L - \lambda_2) - e_L^2}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)}} & -\sqrt{\frac{(a_L - \lambda_3)(c_L - \lambda_3) - e_L^2}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}} \\ \sqrt{\frac{(a_L - \lambda_1)(b_L - \lambda_1) - d_L^2}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)}} & \sqrt{\frac{(a_L - \lambda_2)(b_L - \lambda_2) - d_L^2}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)}} & \sqrt{\frac{(a_L - \lambda_3)(b_L - \lambda_3) - d_L^2}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}} \end{pmatrix}, \tag{49}$$

where λ_i , $i = 1, 2, 3$ are the eigenvalues of $M_e^{(2)T} M_e^{(2)}$ ($M_e^{(2)} M_e^{(2)T}$); $\lambda_1 \equiv m_e^2$, $\lambda_2 \equiv m_\mu^2$, and $\lambda_3 \equiv m_\tau^2$ for charged leptons. A similar expression is obtained for $V_R^{(2)}$ by replacing L by R, where the L and R parameters are defined as

$$M_e^{(2)T} M_e^{(2)} \equiv \begin{pmatrix} a_L & d_L & e_L \\ d_L & b_L & f_L \\ e_L & f_L & c_L \end{pmatrix}, \quad M_e^{(2)} M_e^{(2)T} \equiv \begin{pmatrix} a_R & d_R & e_R \\ d_R & b_R & f_R \\ e_R & f_R & c_R \end{pmatrix}, \quad (50)$$

$$\begin{aligned} a_L &= \Sigma_{11}^2 + \Sigma_{21}^2 + \Sigma_{31}^2, & d_L &= \Sigma_{21}\sqrt{\lambda_-} + \Sigma_{11}\Sigma_{12}, & e_L &= \Sigma_{31}\sqrt{\lambda_+} + \Sigma_{11}\Sigma_{13} \\ b_L &= \lambda_- + \Sigma_{12}^2, & f_L &= \Sigma_{12}\Sigma_{13}, \\ c_L &= \lambda_+ + \Sigma_{13}^2 \end{aligned}, \quad (51)$$

$$\begin{aligned} a_R &= \Sigma_{11}^2 + \Sigma_{12}^2 + \Sigma_{13}^2, & d_R &= \Sigma_{12}\sqrt{\lambda_-} + \Sigma_{11}\Sigma_{21}, & e_R &= \Sigma_{13}\sqrt{\lambda_+} + \Sigma_{11}\Sigma_{31} \\ b_R &= \lambda_- + \Sigma_{21}^2, & f_R &= \Sigma_{21}\Sigma_{31}, \\ c_R &= \lambda_+ + \Sigma_{31}^2 \end{aligned}, \quad (52)$$

and similarly for the u and d quark sectors.

Up to now I have realized a complete exact analytical diagonalization of the resulting charged fermion mass matrices at one and two loops contributions. Thus, from Eqs. (23) and (49) one may obtain exact expressions for the orthogonal matrices

$$V_L \equiv V_L^{(1)} V_L^{(2)}, \quad V_R \equiv V_R^{(1)} V_R^{(2)}. \quad (53)$$

In particular, the V_{CKM} quark mixing matrix (ignoring CP violation) takes the form

$$V_{CKM} = (V_{uL}^{(1)} V_{uL}^{(2)})^T V_{dL}^{(1)} V_{dL}^{(2)} \quad (54)$$

3.3 Approximate mixing matrices for charged fermions

Taking advantage of the strong hierarchy of masses observed in the quarks and charged leptons, it is possible to make good approximations for the mixing matrices $V_L^{(1)}$ and $V_L^{(2)}$ given in the Eqs. (23) and (49), respectively. For instance, the tree level contribution defines the magnitude of the masses for the heaviest fermions m_t, m_b , and m_τ in each sector. One loop contribution determines the masses for m_c, m_s , and m_μ and the mixing angle between the 2 and 3 families, giving simultaneously small corrections to the masses of the heaviest fermions. Finally, the two loop contribution gives masses to the lightest fermions u, d and e and determines their mixing with the families 2 and 3, giving some tiny corrections to the masses of the heavier fermions. I use this perturbative mechanism to make the following approximations:

$$\lambda_- \ll \lambda_+ \quad \text{implies} \quad \frac{D'}{B'^2} = \frac{\lambda_- \lambda_+}{(\lambda_- + \lambda_+)^2} = \frac{\lambda_-}{\lambda_+} \frac{1}{(1 + \frac{\lambda_-}{\lambda_+})^2} \ll 1, \quad (55)$$

and expanding now the square root in Eq.(26), one gets

$$\begin{aligned} \lambda_- &= \frac{B'}{2} \left[1 - \sqrt{1 - 4 \frac{D'}{B'^2}} \right] \approx \frac{D'}{B'} \approx \frac{(a_2 a_3 - a_{23} a_{32})^2}{a_3^2} \\ &= \left(a_2 - \frac{a_{23} a_{32}}{a_3} \right)^2 = (a_2 - \Sigma_{12})^2, \end{aligned} \quad (56)$$

and hence from Eqs. (21) and (27), $\lambda_+ - b'_L = a'_L - \lambda_- \approx a_2^2 + a_{32}^2 - (a_2 - \Sigma_{12})^2 \approx a_{32}^2$. Thus, from Eqs. (23)-(27), the strong hierarchy of masses leads to the approximation

$$V_L^{(1)} \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & -s_{23} \\ 0 & s_{23} & c_{23} \end{pmatrix}, \quad s_{23} \equiv \sqrt{\frac{a_{32}^2}{\lambda_3 - \lambda_2}}, \quad c_{23} = \sqrt{1 - s_{23}^2}, \quad (57)$$

where I have set $\lambda_+ - \lambda_- \approx \lambda_3 - \lambda_2$. Then, in this approach the main contribution to the mixing angle between the families 2 and 3 comes from the mixing matrix element $(V_L^{(1)})_{23}$. So, according to this perturbative approach the two loop contribution $(V_L^{(2)})_{23}$ is negligible, that is, $|(V_L^{(2)})_{23}| \ll |(V_L^{(1)})_{23}|$, and hence it is a good approach to set $(V_L^{(2)})_{23} \approx 0$ in Eq.(49). Let me discuss further about the consistence of this approximation.

From the results of the Appendix, by replacing $a, b, c, d, e, f \rightarrow a_L, b_L, c_L, d_L, e_L, f_L$, one gets the exact equation

$$\Delta_{2L}(\lambda_3) \equiv \lambda_3^2 - (a_L + c_L)\lambda_3 + a_L c_L - e_L^2 = (a_L c_L - e_L^2 - \lambda_1 \lambda_3) - \lambda_3(\lambda_2 - b_L), \quad (58)$$

where $a_L c_L - e_L^2 = (\Sigma_{11}^2 + \Sigma_{21}^2)\lambda_+ - 2\Sigma_{11}\Sigma_{13}\Sigma_{31}\sqrt{\lambda_+} + (\Sigma_{21}^2 + \Sigma_{31}^2)\Sigma_{13}^2 \sim O(2 \text{ loops})^2 \lambda_+$. Let me recall here that from Eq.(31), $\lambda_2 \rightarrow \lambda_-$ and $\lambda_3 \rightarrow \lambda_+$ if one ignores two loop contributions to the masses for the 2 and 3 families. Therefore, because $\lambda_1 \sim O(2 \text{ loops})^2$ and $\lambda_2 - b_L = \lambda_2 - \lambda_- - \Sigma_{12}^2 \sim O(2 \text{ loops})^2$ by construction, one concludes that the right-hand side of Eq. (58) is $\sim O(2 \text{ loops})^2 \lambda_3$, and hence

$$|(V_L^{(2)})_{23}| = \sqrt{\frac{\Delta_{2L}(\lambda_3)}{h(\lambda_3)}} \sim O\left(\frac{2 \text{ loops}}{\sqrt{\lambda_3}}\right) \lesssim O\left(\frac{m_e}{m_\tau}\right) \ll 1. \quad (59)$$

One can also directly write

$$\begin{aligned} \lambda_3^2 - (a_L + c_L)\lambda_3 + a_L c_L - e_L^2 &= \lambda_3^2 - [\lambda_- + \lambda_+ + a_L + \Sigma_{13}^2]\lambda_3 + [\lambda_- + \frac{a_L c_L - e_L^2}{\lambda_3}]\lambda_3 \\ &\approx \lambda_3^2 - [\lambda_- + \lambda_+]\lambda_3 + \lambda_- \lambda_3 \end{aligned} \quad (60)$$

Comparing now Eq. (60) with Eq. (25), one concludes, in what concern the magnitudes for the heaviest fermion in each charged sector; m_t^2 , m_b^2 , and m_τ^2 , that the eigenvalue λ_3 satisfies with good approximation the quadratic equation

$$\Delta_{2L}(\lambda_3) \equiv \lambda_3^2 - (a_L + c_L)\lambda_3 + a_L c_L - e_L^2 \approx 0, \quad (61)$$

and hence $(V_L^{(2)})_{23} \approx 0$ is in good agreement with the radiative corrections. Using now Eq. (61) and the orthogonality of $V_L^{(2)}$, Eq.(49), one reaches the approximation

$$V_L^{(2)} \approx \begin{pmatrix} c_{12}c_{13} & c_{13}s_{12} & -s_{13} \\ -s_{12} & c_{12} & 0 \\ c_{12}s_{13} & s_{12}s_{13} & c_{13} \end{pmatrix} = \begin{pmatrix} c_{13} & 0 & -s_{13} \\ 0 & 1 & 0 \\ s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (62)$$

where the s_{12} and s_{13} mixing angles are identified as

$$\begin{aligned}
s_{12} &\equiv \sqrt{\frac{\Sigma_{21}^2}{\lambda_2 - \lambda_1}} \quad , \quad s_{13} \equiv \sqrt{\frac{\Sigma_{31}^2}{\lambda_3 - \lambda_1}} \\
c_{12} &= \sqrt{1 - s_{12}^2} \quad , \quad c_{13} = \sqrt{1 - s_{13}^2} \quad ,
\end{aligned} \tag{63}$$

and therefore

$$V_L \equiv V_L^{(1)} V_L^{(2)} \approx \begin{pmatrix} c_{12}c_{13} & c_{13}s_{12} & -s_{13} \\ -c_{23}s_{12} - c_{12}s_{13}s_{23} & c_{12}c_{23} - s_{12}s_{13}s_{23} & -c_{13}s_{23} \\ c_{12}c_{23}s_{13} - s_{12}s_{23} & c_{12}s_{23} + c_{23}s_{12}s_{13} & c_{13}c_{23} \end{pmatrix}_L . \tag{64}$$

It is important to emphasize here that Eq. (64) it is not a parametrization but an approximation that is consistent with

- the strong hierarchy of masses for quarks and charged leptons and
- the radiative corrections.

A similar analysis for the R handed mixing matrices yields

$$V_R \equiv V_R^{(1)} V_R^{(2)} \approx \begin{pmatrix} c'_{12}c'_{13} & c'_{13}s'_{12} & -s'_{13} \\ -c'_{23}s'_{12} - c'_{12}s'_{13}s'_{23} & c'_{12}c'_{23} - s'_{12}s'_{13}s'_{23} & -c'_{13}s'_{23} \\ -s'_{12}s'_{23} + c'_{12}c'_{23}s'_{13} & c'_{12}s'_{23} + c'_{23}s'_{12}s'_{13} & c'_{13}c'_{23} \end{pmatrix}_R , \tag{65}$$

where

$$s'_{12} \equiv \sqrt{\frac{\Sigma_{12}^2}{\lambda_2 - \lambda_1}} \quad , \quad s'_{13} \equiv \sqrt{\frac{\Sigma_{13}^2}{\lambda_3 - \lambda_1}} \quad , \quad s'_{23} \equiv \sqrt{\frac{a_{23}^2}{\lambda_3 - \lambda_2}} . \tag{66}$$

So, the strong hierarchy of masses for charged leptons leads to the approach

$$\begin{aligned}
s_{12} &\approx \frac{\Sigma_{21}}{m_\mu} \approx O\left(\frac{2 \text{ loops}}{1 \text{ loop}}\right) \quad , \quad s'_{12} \approx \frac{\Sigma_{12}}{m_\mu} \approx O\left(\frac{2 \text{ loops}}{1 \text{ loop}}\right) , \\
s_{23} &\approx \frac{|a_{32}|}{m_\tau} \approx O\left(\frac{1 \text{ loop}}{\text{tree level}}\right) \quad , \quad s'_{23} \approx \frac{|a_{23}|}{m_\tau} \approx O\left(\frac{1 \text{ loop}}{\text{tree level}}\right) , \\
s_{13} &\approx \frac{|\Sigma_{31}|}{m_\tau} \approx O\left(\frac{2 \text{ loops}}{\text{tree level}}\right) \quad , \quad s'_{13} \approx \frac{|\Sigma_{13}|}{m_\tau} \approx O\left(\frac{2 \text{ loops}}{\text{tree level}}\right) .
\end{aligned} \tag{67}$$

Hence one concludes that the radiative mass mechanism naturally yields the hierarchy

$$s_{13} < s_{12} \quad , \quad s_{23} \quad ; \quad s'_{13} < s'_{12} \quad , \quad s'_{23} . \tag{68}$$

Notice that the approximations realized in this section, Eqs. (57) and (62)-(68), may equally be applied to the e, u, and d charged sectors by replacing properly the mass parameters: $m_\tau^0 \rightarrow m_t^0, m_b^0$, $a_2, a_{23}, a_{32} \rightarrow a_2^{u,d}, a_{23}^{u,d}, a_{32}^{u,d}$ and $\Sigma_{11}, \Sigma_{12}, \Sigma_{13}, \Sigma_{21}, \Sigma_{31} \rightarrow \Sigma_{11}^{u,d}, \Sigma_{12}^{u,d}, \Sigma_{13}^{u,d}, \Sigma_{21}^{u,d}, \Sigma_{31}^{u,d}$, respectively, in an obvious notation.

4 Quantitative analysis of masses and mixing for charged leptons

Using the orthogonality conditions of U and Eq.(171) of the Appendix, one can compute the parameters involved in the charged lepton mass matrices; one gets explicitly

$$\begin{aligned}
a_2 &= m_\tau^0 \frac{Y_{23}^2}{16\pi^2} \sum_{k=1}^4 U_{1k} U_{4k} \ln \frac{M_k^2}{m_\tau^2} = m_\tau^0 \frac{Y_{23}^2}{16\pi^2} \sum_{k=1}^4 \frac{f_4(\eta_k)}{h(\eta_k)} \ln \frac{\eta_k}{m_\tau^2} \equiv m_\tau^0 \frac{Y_{23}^2}{16\pi^2} F_{22} \\
a_{23} &= m_\tau^0 \frac{Y_{23} Y_{33}}{16\pi^2} \sum_{k=1}^4 U_{2k} U_{4k} \ln \frac{M_k^2}{m_\tau^2} = m_\tau^0 \frac{Y_{23} Y_{33}}{16\pi^2} \sum_{k=1}^4 \frac{g_4(\eta_k)}{h(\eta_k)} \ln \frac{\eta_k}{m_\tau^2} \equiv -m_\tau^0 \frac{Y_{23} Y_{33}}{16\pi^2} F_{23} \\
a_{32} &= m_\tau^0 \frac{Y_{23} Y_{33}}{16\pi^2} \sum_{k=1}^4 U_{1k} U_{3k} \ln \frac{M_k^2}{m_\tau^2} = m_\tau^0 \frac{Y_{23} Y_{33}}{16\pi^2} \sum_{k=1}^4 \frac{f_3(\eta_k)}{h(\eta_k)} \ln \frac{\eta_k}{m_\tau^2} \equiv -m_\tau^0 \frac{Y_{23} Y_{33}}{16\pi^2} F_{32} \\
\sigma &= \frac{Y_{12}^2}{16\pi^2} \sum_{k=1}^4 U_{2k} U_{3k} \ln \frac{M_k^2}{m_\tau^2} = \frac{Y_{12}^2}{16\pi^2} \sum_{k=1}^4 \frac{g_3(\eta_k)}{h(\eta_k)} \ln \frac{\eta_k}{m_\tau^2} \equiv \frac{Y_{12}^2}{16\pi^2} F_\sigma
\end{aligned} \tag{69}$$

where $\eta_k \equiv M_k^2$, and the dimensionless functions F_{22} , F_{23} , F_{32} , and F_σ may be written as

$$\begin{aligned}
F_{22} &= \sum_{k=2}^4 \frac{f_4(\eta_k)}{h(\eta_k)} \ln \frac{\eta_k}{\eta_1} = \frac{bcd}{|h(\eta_2)|} \ln \frac{\eta_2}{\eta_1} - \frac{bcd}{h(\eta_3)} \ln \frac{\eta_3}{\eta_1} + \frac{bcd}{|h(\eta_4)|} \ln \frac{\eta_4}{\eta_1} > 0 \\
F_{23} &= -\sum_{k=2}^4 \frac{g_4(\eta_k)}{h(\eta_k)} \ln \frac{\eta_k}{\eta_1} = \frac{cd(a_1-\eta_2)}{|h(\eta_2)|} \ln \frac{\eta_2}{\eta_1} - \frac{cd(a_1-\eta_3)}{h(\eta_3)} \ln \frac{\eta_3}{\eta_1} + \frac{cd(a_1-\eta_4)}{|h(\eta_4)|} \ln \frac{\eta_4}{\eta_1} > 0 \\
F_{32} &= -\sum_{k=2}^4 \frac{f_3(\eta_k)}{h(\eta_k)} \ln \frac{\eta_k}{\eta_1} = \frac{bc(a_4-\eta_2)}{|h(\eta_2)|} \ln \frac{\eta_2}{\eta_1} - \frac{bc(a_4-\eta_3)}{h(\eta_3)} \ln \frac{\eta_3}{\eta_1} + \frac{bc(a_4-\eta_4)}{|h(\eta_4)|} \ln \frac{\eta_4}{\eta_1} > 0 \quad , \\
F_\sigma &= \sum_{k=2}^4 \frac{g_3(\eta_k)}{h(\eta_k)} \ln \frac{\eta_k}{\eta_1} = \frac{c(a_1-\eta_2)(a_4-\eta_2)}{|h(\eta_2)|} \ln \frac{\eta_2}{\eta_1} - \frac{c(a_1-\eta_3)(a_4-\eta_3)}{h(\eta_3)} \ln \frac{\eta_3}{\eta_1} \\
&\quad + \frac{c(a_1-\eta_4)(a_4-\eta_4)}{|h(\eta_4)|} \ln \frac{\eta_4}{\eta_1} > 0
\end{aligned} \tag{70}$$

where

$$|h(\eta_2)| \equiv (\eta_2 - \eta_1)(\eta_3 - \eta_2)(\eta_4 - \eta_2) \quad , \quad |h(\eta_4)| \equiv (\eta_4 - \eta_1)(\eta_4 - \eta_2)(\eta_4 - \eta_3) . \tag{71}$$

To leading order in the radiative loop corrections, one reaches the approximations

$$\begin{aligned}
m_\tau &\equiv \sqrt{\lambda_3} \approx \sqrt{\lambda_+} \approx a_3 = m_\tau^0 \quad , \\
m_\mu &\equiv \sqrt{\lambda_2} \approx \sqrt{\lambda_-} \approx a_2 \quad , \\
m_e &\equiv \sqrt{\lambda_1} \approx \Sigma_{11} = a_2 \sigma \approx m_\mu \sigma \quad ,
\end{aligned} \tag{72}$$

and hence from Eqs. (69)-(72):

$$\begin{aligned}
\frac{a_2}{m_\tau^0} &= \frac{Y_{23}^2}{16\pi^2} F_{22} \approx \frac{m_\mu}{m_\tau} \quad ; \quad \sigma = \frac{Y_{12}^2}{16\pi^2} F_\sigma \approx \frac{m_e}{m_\mu} \\
\text{or} \quad \frac{Y_{23}}{4\pi} &\approx \frac{.243842}{\sqrt{F_{22}}} \quad ; \quad \frac{Y_{12}}{4\pi} \approx \frac{.0695437}{\sqrt{F_\sigma}} .
\end{aligned} \tag{73}$$

So, in this approach the following relationships hold:

$$\left(\frac{Y_{12}Y_{23}}{16\pi^2}\right)^2 \approx \frac{\frac{m_e}{m_\tau}}{F_{22} F_\sigma} = \frac{2.875643 \times 10^{-4}}{F_{22} F_\sigma}, \quad (74)$$

$$\frac{Y_{12}}{Y_{23}} \approx (.285199) \frac{\sqrt{F_{22}}}{\sqrt{F_\sigma}}, \quad (75)$$

$$\frac{|a_{23}|}{m_\mu} \approx \frac{|a_{23}|}{a_2} = \frac{Y_{33}F_{23}}{Y_{23}F_{22}} = c_\alpha \frac{Y_{12}F_{23}}{Y_{23}F_{22}} = c_\alpha (.285199) \frac{F_{23}}{\sqrt{F_{22}F_\sigma}}, \quad (76)$$

$$\frac{|a_{32}|}{m_\mu} \approx \frac{|a_{32}|}{a_2} = \frac{Y_{33}F_{32}}{Y_{23}F_{22}} = c_\alpha \frac{Y_{12}F_{32}}{Y_{32}F_{22}} = c_\alpha (.285199) \frac{F_{32}}{\sqrt{F_{22}F_\sigma}}$$

$$\Sigma_{12} = \Sigma_{21} = \frac{1}{c_\alpha} \frac{a_{23}a_{32}}{a_3} = \frac{1}{c_\alpha} \frac{|a_{23}||a_{32}|}{m_\tau^0} \approx \frac{1}{c_\alpha} |a_{23}| s_{23}^e = \frac{1}{c_\alpha} |a_{32}| s_{23}'^e, \quad (77)$$

$$|\Sigma_{31}| = c_\alpha |a_{32}| \sigma + \frac{1}{c_\alpha} \frac{a_2 |a_{23}|}{a_3} \approx c_\alpha \frac{m_e}{m_\mu} |a_{32}| + \frac{1}{c_\alpha} \frac{m_\mu}{m_\tau} |a_{23}|, \quad (78)$$

$$|\Sigma_{13}| = c_\alpha |a_{23}| \sigma + \frac{1}{c_\alpha} \frac{a_2 |a_{32}|}{a_3} \approx c_\alpha \frac{m_e}{m_\mu} |a_{23}| + \frac{1}{c_\alpha} \frac{m_\mu}{m_\tau} |a_{32}|$$

where the superscript e denotes the charged lepton sector. Therefore, the mixing angles in (V_{eL}) , Eq. (64), and (V_{eR}) , Eq. (65), may be expressed as

$$\begin{aligned} s_{23}^e &\approx \sqrt{\frac{a_{32}^2}{\lambda_3 - \lambda_2}} \approx \frac{|a_{32}|}{m_\tau} = c_\alpha \left(\frac{Y_{12}Y_{23}}{16\pi^2}\right) F_{32} = c_\alpha (0.016957) \frac{F_{32}}{\sqrt{F_{22}F_\sigma}} \\ s_{12}^e &\approx \sqrt{\frac{\Sigma_{21}^2}{\lambda_2 - \lambda_1}} \approx \frac{\Sigma_{21}}{m_\mu} = \frac{1}{c_\alpha} \frac{|a_{23}|}{m_\mu} s_{23}^e \approx (.285199) \frac{F_{23}}{\sqrt{F_{22}F_\sigma}} s_{23}^e, \\ s_{13}^e &\approx \sqrt{\frac{\Sigma_{31}^2}{\lambda_3 - \lambda_1}} \approx \frac{|\Sigma_{31}|}{m_\tau} \approx c_\alpha \frac{m_e}{m_\mu} s_{23}^e + \left(\frac{m_\mu}{m_\tau}\right)^2 \frac{s_{12}^e}{s_{23}^e} \end{aligned} \quad (79)$$

and

$$\begin{aligned} s_{23}'^e &\approx \sqrt{\frac{a_{23}^2}{\lambda_3 - \lambda_2}} \approx \frac{|a_{23}|}{m_\tau} = c_\alpha \left(\frac{Y_{12}Y_{23}}{16\pi^2}\right) F_{23} = c_\alpha (.016957) \frac{F_{23}}{\sqrt{F_{22}F_\sigma}} \\ s_{12}'^e &\approx \sqrt{\frac{\Sigma_{12}^2}{\lambda_2 - \lambda_1}} \approx \frac{\Sigma_{12}}{m_\mu} = \frac{1}{c_\alpha} \frac{|a_{32}|}{m_\mu} s_{23}'^e \approx (.285199) \frac{F_{32}}{\sqrt{F_{22}F_\sigma}} s_{23}'^e, \\ s_{13}'^e &\approx \sqrt{\frac{\Sigma_{13}^2}{\lambda_3 - \lambda_1}} \approx \frac{|\Sigma_{13}|}{m_\tau} \approx c_\alpha \frac{m_e}{m_\mu} s_{23}'^e + \left(\frac{m_\mu}{m_\tau}\right)^2 \frac{s_{12}'^e}{s_{23}'^e} \end{aligned} \quad (80)$$

respectively, with the relations

$$s_{12}^e = s_{12}'^e \quad \text{and hence} \quad \frac{s_{23}^e}{s_{23}'^e} = \frac{|a_{32}|}{|a_{23}|} = \frac{F_{32}}{F_{23}}. \quad (81)$$

4.1 Upper bounds for charged lepton mixing angles

Each particular set of scalar mass parameters in Eq.(165) define a spectrum of scalar mass eigenvalues $\eta_1, \eta_2, \eta_3, \eta_4$, the values for F_{22}, F_{23}, F_{32} and F_σ through the Eq.(70), as well as the magnitudes for mixing angles in V_{eL} and V_{eR} through Eqs. (79) and (80). A numerical evaluation shows that the variation of these mixing angles is relatively small for a large region in the space mass parameters. So, in order to find out the orders of magnitude for these mixing angles, let me redefine the parameters of M_ϕ^2 , Eq.(165), in such a way

$$M_\phi^2 = \begin{pmatrix} a'_1 & b' & 0 & 0 \\ b' & a'_2 & c' & 0 \\ 0 & c' & a'_3 & d' \\ 0 & 0 & d' & a'_4 \end{pmatrix} M^2, \quad (82)$$

where $a'_1, a'_2, a'_3, a'_4, b', c'$ and d' are positive real numbers, while M is a mass parameter in the TeV region. Mixing angles in V_{eL} and V_{eR} do not depend on M . The value of M^2 may be determined for instance by specifying the value of the lightest scalar mass eigenvalue $\eta_1 \equiv M_1^2$.

Setting for example and simplicity $a'_1 = a'_4, b' = c' = d'$ one gets $F_{23} = F_{32}$ and hence $s_{23}^e = s_{23}'^e, s_{12}^e = s_{12}'^e, s_{13}^e = s_{13}'^e$. So, in the simplified parameter space region defined by $a'_1 = a'_4 = 1, 2 \leq a'_2 \leq 120, 3 \leq a'_3 \leq 125, 1 \leq b' = c' = d' \leq 10$, one gets the following range of values for mixing angles in the charged lepton sector:

$$\begin{aligned} 4.666163 \times 10^{-3} &\lesssim s_{23}^e = s_{23}'^e \lesssim 1.34417 \times 10^{-2}, \\ 0.105688 &\lesssim \frac{s_{12}^e}{s_{23}^e} = \frac{s_{12}'^e}{s_{23}'^e} \lesssim 0.240805, \\ 4.931603 \times 10^{-4} &\lesssim s_{12}^e = s_{12}'^e \lesssim 3.236832 \times 10^{-3}, \\ 3.904076 \times 10^{-4} &\lesssim s_{13}^e = s_{13}'^e \lesssim 9.123709 \times 10^{-4}, \end{aligned} \quad (83)$$

where the upper and lower bounds are obtained with the values $a'_2 = 2, a'_3 = 3, b' = c' = d' = 1$, and $a'_2 = 120, a'_3 = 125, b' = c' = d' = 10$, respectively, and where I have used the range of values $c_\alpha = \frac{Y_{33}}{Y_{12}} = \frac{c}{2d} = 0.742528 - 0.938792$ corresponding to the global parameter space region defined by Eq. (103) in the analysis of neutrino mixing.

5 Neutrino masses and V_{PMNS}

From the Yukawa couplings of Eq.(7), the mass matrix for the left-handed neutrinos is obtained as

$$M_\nu = \begin{pmatrix} 0 & Y_{12} v_{10}/2 & 0 \\ Y_{12} v_{10}/2 & 0 & Y_{23} v_9/2 \\ 0 & Y_{23} v_9/2 & Y_{33} v_{10} \end{pmatrix} \equiv \begin{pmatrix} 0 & d & 0 \\ d & 0 & f \\ 0 & f & c \end{pmatrix}. \quad (84)$$

One may diagonalize M_ν as

$$U_\nu^T M_\nu U_\nu = M_\nu^d, \quad (85)$$

where $M_\nu^d \equiv \text{diag}(\xi_1, \xi_2, \xi_3)$ is the diagonal matrix with ξ_1 , ξ_2 , and ξ_3 being the eigenvalues of M_ν , and U_ν is the rotation matrix which connects the gauge states with the corresponding eigenstates.

The eigenvalues ξ_1 , ξ_2 , and ξ_3 satisfy the following nonlinear relationships with the parameters d , f and c of M_ν , Eq.(84):

$$\begin{aligned} \xi_1 + \xi_2 + \xi_3 &= c \\ \xi_1 \xi_2 + \xi_1 \xi_3 + \xi_2 \xi_3 &= -d^2 - f^2 \\ \xi_1 \xi_2 \xi_3 &= -d^2 c \end{aligned} \quad (86)$$

The square matrix elements $U_{\nu ij}^2$ may be obtained from those of $(V_L^{(2)})_{ij}^2$, Eq.(49), by replacing $a_L, b_L, c_L, d_L, e_L, f_L \rightarrow 0, 0, c, d, 0, f$ and $\lambda_i \rightarrow \xi_i$ respectively. However, from the Eq.(86) and assuming $c, d, f > 0$, it is easy to conclude that one of the ξ_i , $i = 1, 2, 3$ is negative. Thus, the eigenvalues ξ_i cannot be directly associated to the physical neutrino masses.

Setting $\xi_3 > 0$ and computing explicitly the $U_{\nu ij}^2$ elements, one arrives to the following statements:

1. Assuming normal hierarchy

$$\xi_1^2 < \xi_2^2 < \xi_3^2 \text{ implies } \xi_1 > 0 \text{ and } \xi_2 < 0, \quad (87)$$

2. Assuming the hierarchy

$$\xi_2^2 < \xi_1^2 < \xi_3^2 \text{ implies } \xi_1 < 0 \text{ and } \xi_2 > 0 \quad (88)$$

In what follows, I assume a normal hierarchy for the squared eigenvalues ξ_i^2 as in Eq.(87)². In the literature there exists a lot of models dealing with normal neutrino mass hierarchy[9]. Now I define the unitary matrix $V_\nu \equiv U_\nu \text{diag}(1, i, 1)$ [10], which may be written as

$$V_\nu = \begin{pmatrix} \sqrt{\frac{f^2 m_2 m_3}{(m_2^2 - m_1^2)(m_3^2 - m_1^2)}} & -i \sqrt{\frac{f^2 m_1 m_3}{(m_2^2 - m_1^2)(m_3^2 - m_2^2)}} & \sqrt{\frac{f^2 m_1 m_2}{(m_3^2 - m_1^2)(m_3^2 - m_2^2)}} \\ \sqrt{\frac{f^2 c m_1}{(m_2^2 - m_1^2)(m_3^2 - m_1^2)}} & i \sqrt{\frac{f^2 c m_2}{(m_2^2 - m_1^2)(m_3^2 - m_2^2)}} & \sqrt{\frac{f^2 c m_3}{(m_3^2 - m_1^2)(m_3^2 - m_2^2)}} \\ -\sqrt{\frac{(d^2 - m_1^2)(d^2 + f^2 - m_1^2)}{(m_2^2 - m_1^2)(m_3^2 - m_1^2)}} & -i \sqrt{\frac{(m_2^2 - d^2)(d^2 + f^2 - m_2^2)}{(m_2^2 - m_1^2)(m_3^2 - m_2^2)}} & \sqrt{\frac{(m_3^2 - d^2)(m_3^2 - d^2 - f^2)}{(m_3^2 - m_1^2)(m_3^2 - m_2^2)}} \end{pmatrix}, \quad (89)$$

²The second possibility, Eq.(88), does not change any conclusion about this model

or equivalently in the form

$$V_\nu = \begin{pmatrix} \sqrt{\frac{m_2 m_3 (m_3 - m_2)}{c(m_1 + m_2)(m_3 - m_1)}} & -i\sqrt{\frac{m_1 m_3 (m_1 + m_3)}{c(m_1 + m_2)(m_3 + m_2)}} & \sqrt{\frac{m_1 m_2 (m_2 - m_1)}{c(m_3 - m_1)(m_3 + m_2)}} \\ \sqrt{\frac{m_1 (m_3 - m_2)}{(m_1 + m_2)(m_3 - m_1)}} & i\sqrt{\frac{m_2 (m_1 + m_3)}{(m_1 + m_2)(m_3 + m_2)}} & \sqrt{\frac{m_3 (m_2 - m_1)}{(m_3 - m_1)(m_3 + m_2)}} \\ -\sqrt{\frac{m_1 (m_1 + m_3)(m_2 - m_1)}{c(m_1 + m_2)(m_3 - m_1)}} & -i\sqrt{\frac{m_2 (m_3 - m_2)(m_2 - m_1)}{c(m_1 + m_2)(m_3 + m_2)}} & \sqrt{\frac{m_3 (m_1 + m_3)(m_3 - m_2)}{c(m_3 - m_1)(m_3 + m_2)}} \end{pmatrix} \quad (90)$$

after using properly the results of the Appendix and making the identification

$$(\xi_1, -\xi_2, \xi_3) = (m_1, m_2, m_3) \quad (91)$$

between the eigenvalues ξ_i and the physical neutrino masses m_1 , m_2 , and m_3 . Therefore, for neutrinos the transformation between gauge, $\psi_{\nu L}^{0T} = (\nu_e^0, \nu_\mu^0, \nu_\tau^0)_L$, and mass eigenstates, $\psi_{\nu L}^T = (\nu_1, \nu_2, \nu_3)_L$, is

$$\psi_{\nu L}^0 = V_\nu \psi_{\nu L} . \quad (92)$$

From Eq.(91) and the definition of V_ν it is easy to verify that

$$V_\nu^T M_\nu V_\nu = \text{diag}(m_1, m_2, m_3) , \quad (93)$$

and hence one may write Eq.(86) in terms of neutrino masses:

$$\begin{aligned} m_1 + m_3 - m_2 &= c , \\ m_1 m_2 - m_1 m_3 + m_2 m_3 &= d^2 + f^2 , \\ m_1 m_2 m_3 &= d^2 c . \end{aligned} \quad (94)$$

The combination of these relationships yields the useful equality

$$f^2 c = (m_3 - m_2)(m_3 + m_1)(m_2 - m_1) . \quad (95)$$

Notice that Eqs. (94) and (95) allow one to write all the matrix elements $(V_\nu)_{ij}$ completely in terms of the physical neutrino masses m_1 , m_2 , and m_3 as in Eq.(90).

5.1 V_{PMNS} lepton mixing matrix

The current experimental study of neutrino oscillation phenomena gives as a result that in the lepton sector the mixing matrix V_{PMNS} behaves close to the so-called "tribimaximal mixing" (TBM)[11]. In particular, according to Eq.(1), the mixing angles θ_{12} and θ_{23} are large, $\sin \theta_{12}$ and $\sin \theta_{23} \lesssim O(1)$, while θ_{13} has not yet been measured. So, taking the ranges of values in Eq.(83) as the typical orders of magnitude for the mixing angles in the charged lepton sector, it is then clear that mixing in the lepton sector should come almost completely from neutrino mixing, and then one may approach with good precision

$$V_{PMNS} \equiv (V_{eL})^\dagger V_\nu \approx V_\nu . \quad (96)$$

Thus, from Eqs. (89), (90), and (96), the V_{PMNS} lepton mixing matrix in this model may be approached as

$$V_{PMNS} \approx \begin{pmatrix} c_{12}c_{13} & -i c_{13}s_{12} & s_{13} \\ c_{23}s_{12} - c_{12}s_{23}s_{13} & i (c_{12}c_{23} + s_{12}s_{23}s_{13}) & c_{13}s_{23} \\ -(s_{12}s_{23} + c_{12}c_{23}s_{13}) & -i (c_{12}s_{23} - c_{23}s_{12}s_{13}) & c_{13}c_{23} \end{pmatrix}, \quad (97)$$

where the lepton mixing angles are identified as

$$\begin{aligned} S_{13}^2 &\equiv (V_\nu)_{13}^2 = \frac{f^2 m_1 m_2}{(m_3^2 - m_1^2)(m_3^2 - m_2^2)} = \frac{m_1 m_2 (m_2 - m_1)}{(m_1 + m_3 - m_2)(m_3 - m_1)(m_3 + m_2)}, \\ S_{12}^2 &\equiv \frac{m_3^2 - m_1^2}{m_2^2 - m_1^2} \frac{f^2 m_1 m_3}{(m_3^2 - m_1^2)(m_3^2 - m_2^2) - f^2 m_1 m_2} = \frac{s_{13}^2 m_3 m_3^2 - m_1^2}{c_{13}^2 m_2 m_2^2 - m_1^2}, \\ S_{23}^2 &\equiv \frac{f^2 c m_3}{(m_3^2 - m_1^2)(m_3^2 - m_2^2) - f^2 m_1 m_2} = \frac{s_{13}^2 m_3 c}{c_{13}^2 m_2 m_1}, \end{aligned} \quad (98)$$

and

$$\begin{aligned} c_{13}^2 &= 1 - s_{13}^2 = \frac{(m_3^2 - m_1^2)(m_3^2 - m_2^2) - f^2 m_1 m_2}{(m_3^2 - m_1^2)(m_3^2 - m_2^2)}, \\ c_{23}^2 &= 1 - s_{23}^2 = \frac{1}{c_{13}^2} \frac{m_3}{c} \frac{(m_3 + m_1)(m_3 - m_2)}{(m_3 - m_1)(m_3 + m_2)}, \\ c_{12}^2 &= 1 - s_{12}^2 = \frac{s_{13}^2 m_3 (m_3^2 - m_2^2)}{c_{13}^2 m_1 (m_2^2 - m_1^2)}. \end{aligned} \quad (99)$$

The combination of the last two equations yields

$$\begin{aligned} \sin^2 2\theta_{12} &= 4 \frac{m_1 m_2}{(m_1 + m_2)^2} \frac{(m_3^2 - m_1^2)(m_3^2 - m_2^2)}{(m_3^2 - m_2^2 - m_1^2 + m_1 m_2)^2}, \\ \sin^2 2\theta_{23} &= 4 \frac{(m_1 + m_3 - m_2)(m_2 - m_1)(m_3 - m_2)(m_3 + m_1)}{(m_3^2 - m_2^2 - m_1^2 + m_1 m_2)^2}. \end{aligned} \quad (100)$$

5.1.1 Numerical analysis

It is clear from Eqs. (89), (90), and (97)-(100) that this model predicts $s_{13}^2 > 0$, implying some deviation from the TBM limit. The allowed range of values for lepton mixing depends on the value, or range of values, used for s_{13} . To perform a numerical analysis let me introduce the parameters k and x defined as

$$k \equiv \frac{m_3 - m_2}{m_1}, \quad x \equiv \frac{m_2}{m_1} > 1. \quad (101)$$

One may write all the matrix elements of V_{PMNS} in terms of these two parameters. In particular s_{13}^2 may be expressed as

$$s_{13}^2 = \frac{x(x-1)}{(k+1)(k+x-1)(k+2x)} . \quad (102)$$

The last equation may be used now to invert x in terms of s_{13}^2 and k , and thus $\sin^2 2\theta_{12}$ and $\sin^2 2\theta_{23}$ may be written in terms of s_{13}^2 and k . A numerical analysis shows that for the range of values $0.033 \leq s_{13}^2 \leq 0.04$ one may obtain large mixing angles for θ_{12} and θ_{23} within the allowed limits of Eq.(1). This region for s_{13}^2 is consistent with the upper bound provided by the CHOOZ experiment [12] ($s_{13}^2 \lesssim 0.04$). I point out the allowed magnitudes for lepton mixing in the following (s_{13}^2, k) parameter space regions:

$$\text{global parameter space:} \quad 0.033 \leq s_{13}^2 \leq 0.04 \quad , \quad 3.2 \leq k \leq 4.1 \quad (103)$$

This region yields the range of mixing angles:

$$0.64865 \leq \sin^2 2\theta_{12} \leq 0.818112 \quad , \quad 0.813749 \leq \sin^2 2\theta_{23} \leq 0.919788 , \quad (104)$$

and the V_{PMNS} unitary mixing matrix with the following range of magnitudes:

$$V_{PMNS} \approx \begin{pmatrix} 0.830485 - 0.874368 & 0.442132 - 0.526588 & 0.181659 - 0.2 \\ 0.254605 - 0.371263 & 0.765592 - 0.773352 & 0.524249 - 0.586563 \\ 0.401143 - 0.432288 & 0.367937 - 0.461950 & 0.784821 - 0.831963 \end{pmatrix} . \quad (105)$$

Recall that the above range of values is restricted by the constraints imposed by the unitarity of V_{PMNS} ; that is, choosing a specific value of one entry further restricts the range of values for the other entries. It is clear from Eq.(104) that only part of the values in Eq.(105) are within the allowed limits of Eq.(1). Given a particular value for s_{13}^2 in Eq.(103), it is possible to specify the k parameter region where lepton mixing lies within these allowed limits. I point out below these range of values for $s_{13}^2 = 0.034$, 0.037 , and 0.04 , respectively:

$$\begin{aligned} \text{Case A:} \quad & s_{13}^2 = 0.034 \quad , \quad 3.88182 \leq k \leq 4.02591 \quad (4.50978 \leq x \leq 4.79497) \\ & 0.7 \leq \sin^2 2\theta_{12} \leq 0.719315 \quad , \quad 0.87 \leq \sin^2 2\theta_{23} \leq 0.878086 \end{aligned} \quad (106)$$

$$V_{PMNS} \approx \begin{pmatrix} 0.859588 - 0.864610 & 0.467386 - 0.476558 & 0.184390 \\ 0.298043 - 0.308732 & 0.771902 - 0.772539 & 0.555744 - 0.560673 \\ 0.404500 - 0.407176 & 0.420784 - 0.429807 & 0.807246 - 0.810647 \end{pmatrix} \quad (107)$$

$$\begin{aligned} \text{Case B:} \quad & s_{13}^2 = 0.037 \quad , \quad 3.52059 \leq k \leq 3.8732 \quad (4.18727 \leq x \leq 4.91525) \\ & 0.7 \leq \sin^2 2\theta_{12} \leq 0.749742 \quad , \quad 0.87 \leq \sin^2 2\theta_{23} \leq 0.890806 \end{aligned} \quad (108)$$

$$V_{PMNS} \approx \begin{pmatrix} 0.849926 - 0.863266 & 0.466660 - 0.490536 & 0.192353 \\ 0.289950 - 0.318087 & 0.768718 - 0.770414 & 0.554881 - 0.567795 \\ 0.413159 - 0.420055 & 0.410422 - 0.434385 & 0.800381 - 0.809387 \end{pmatrix} \quad (109)$$

$$\begin{aligned} \text{Case C:} \quad & s_{13}^2 = 0.04 \quad , \quad 3.21323 \leq k \leq 3.73671 \quad (3.91261 \leq x \leq 5.03901) \\ & 0.7 \leq \sin^2 2\theta_{12} \leq 0.777209 \quad , \quad 0.87 \leq \sin^2 2\theta_{23} \leq 0.902305 \end{aligned} \quad (110)$$

$$V_{PMNS} \approx \begin{pmatrix} 0.840573 - 0.861920 & 0.465933 - 0.503425 & 0.2 \\ 0.282093 - 0.326762 & 0.765697 - 0.768410 & 0.554016 - 0.57443 \\ 0.421328 - 0.432045 & 0.400339 - 0.438694 & 0.793744 - 0.808125 \end{pmatrix} \quad (111)$$

An additional analysis shows that for $0.0375 \lesssim s_{13}^2 \lesssim 0.04$ and $0.035 \lesssim s_{13}^2 \lesssim 0.04$ one may specify a k parameter region where lepton mixing lies within the 3σ allowed ranges reported in Refs.[13, 14], respectively.

5.1.2 Neutrino masses

With the purpose to obtain some rough estimation for the order of magnitudes of neutrino masses let me use the range of values for lepton mixing in Eqs. (106)-(111) and the bounds for Δm_{sol}^2 and Δm_{atm}^2 of Eq. (1). One gets the following neutrino masses.

$$\Delta m_{\text{sol}}^2 = m_2^2 - m_1^2 = (x^2 - 1) m_1^2:$$

$$\begin{aligned} m_1 &\approx (1.796 - 2.145 \quad , \quad 1.750 - 2.320 \quad , \quad 1.706 - 2.494) \times 10^{-3} \text{ eV} \\ m_2 &\approx (8.103 - 10.28 \quad , \quad 7.331 - 11.40 \quad , \quad 6.675 - 12.56) \times 10^{-3} \text{ eV} \quad , \\ m_3 &\approx (1.507 - 1.892 \quad , \quad 1.349 - 2.039 \quad , \quad 1.215 - 2.188) \times 10^{-2} \text{ eV} \end{aligned} \quad (112)$$

where the first, second, and third range of values for each m_i , $i = 1, 2, 3$ correspond to $s_{13}^2 = 0.034$, $s_{13}^2 = 0.037$, and $s_{13}^2 = 0.04$ respectively.

$$\Delta m_{\text{atm}}^2 = m_3^2 - m_2^2 = k(k + 2x) m_1^2:$$

$$\begin{aligned} m_1 &\approx (5.053 - 8.117 \quad , \quad 5.135 - 8.876 \quad , \quad 5.207 - 9.645) \times 10^{-3} \text{ eV} \\ m_2 &\approx (2.279 - 3.892 \quad , \quad 2.150 - 4.363 \quad , \quad 2.037 - 4.860) \times 10^{-2} \text{ eV} \\ m_3 &\approx (4.240 - 7.160 \quad , \quad 3.958 - 7.801 \quad , \quad 3.710 - 8.464) \times 10^{-2} \text{ eV} \end{aligned} \quad (113)$$

6 Quantitative analysis of quark masses and V_{CKM}

To leading order in the radiative loop corrections, one gets the approximations

$$\begin{aligned} m_b &\equiv \sqrt{\lambda_3^d} \approx \sqrt{\lambda_+^d} \approx a_3^d = m_b^0 \quad , \quad m_t \equiv \sqrt{\lambda_3^u} \approx \sqrt{\lambda_+^u} \approx a_3^u = m_t^0 \\ m_s &\equiv \sqrt{\lambda_2^d} \approx \sqrt{\lambda_-^d} \approx a_2^d \quad , \quad m_c \equiv \sqrt{\lambda_2^u} \approx \sqrt{\lambda_-^u} \approx a_2^u \\ m_d &\equiv \sqrt{\lambda_1^d} \approx \Sigma_{11}^d = a_2^d \sigma^d \approx m_s \sigma^d \quad , \quad m_u \equiv \sqrt{\lambda_1^u} \approx \Sigma_{11}^u = a_2^u \sigma^u \approx m_c \sigma^u \end{aligned} \quad (114)$$

and then one obtains the relations

$$\begin{aligned}\sigma^d &= \frac{Y_{12}^q Y_{12}^d}{16\pi^2} F_\sigma^d \approx \frac{m_d}{m_s} \quad , \quad \sigma^u = \frac{Y_{12}^q Y_{12}^u}{16\pi^2} F_\sigma^u \approx \frac{m_u}{m_c} \\ \frac{a_2^d}{m_b^0} &= \frac{Y_{23}^q Y_{23}^d}{16\pi^2} F_{22}^d \approx \frac{m_s}{m_b} \quad , \quad \frac{a_2^u}{m_t^0} = \frac{Y_{23}^q Y_{23}^u}{16\pi^2} F_{22}^u \approx \frac{m_c}{m_t}\end{aligned}\tag{115}$$

where the functions $F_{22}^{d,u}$, $F_{23}^{d,u}$, $F_{32}^{d,u}$, $F_\sigma^{d,u}$ are defined analogous to those for the charged lepton sector in Eq.(70).

Hence the mixing angles for the d and u quark sectors, V_L^d and V_L^u , Eq.(64), may be approximated as

$$\begin{aligned}s_{23}^d &\approx \frac{Y_{33}^d}{Y_{23}^d} \frac{F_{32}^d}{F_{22}^d} \frac{m_s}{m_b} \quad , \quad s_{23}^u \approx \frac{Y_{33}^u}{Y_{23}^u} \frac{F_{32}^u}{F_{22}^u} \frac{m_c}{m_t} \\ s_{12}^d &\approx \frac{Y_{12}^q}{Y_{23}^q} \frac{F_{23}^d}{F_{22}^d} s_{23}^d \quad , \quad s_{12}^u \approx \frac{Y_{12}^q}{Y_{23}^q} \frac{F_{23}^u}{F_{22}^u} s_{23}^u \\ s_{13}^d &\approx \frac{Y_{33}^d}{Y_{12}^d} \frac{m_d}{m_s} s_{23}^d + \left(\frac{m_s}{m_b}\right)^2 \frac{s_{12}^d}{s_{23}^d} \quad , \quad s_{13}^u \approx \frac{Y_{33}^u}{Y_{12}^u} \frac{m_u}{m_c} s_{23}^u + \left(\frac{m_c}{m_t}\right)^2 \frac{s_{12}^u}{s_{23}^u}\end{aligned}\tag{116}$$

6.1 Numerical analysis

To explore the allowed magnitudes for mixing angles in V_{CKM} and without lost of generality, let me assume for simplicity the relationships

$$F_{22}^d = F_{22}^u \equiv \mathcal{F}_{22} \quad , \quad F_{23}^d = F_{23}^u \equiv \mathcal{F}_{23} \quad , \quad F_{32}^d = F_{32}^u \equiv \mathcal{F}_{32} \quad , \quad F_\sigma^d = F_\sigma^u \equiv \mathcal{F}_\sigma\tag{117}$$

From Eqs. (114)-(117) one gets the following useful relationships to hold

$$\begin{aligned}\frac{Y_{12}^q}{Y_{23}^q} \frac{Y_{12}^d}{Y_{23}^d} \frac{\mathcal{F}_\sigma}{\mathcal{F}_{22}} &\approx \frac{m_d}{m_s^2} \quad , \quad \frac{Y_{12}^q}{Y_{23}^q} \frac{Y_{12}^u}{Y_{23}^u} \frac{\mathcal{F}_\sigma}{\mathcal{F}_{22}} \approx \frac{m_u}{m_c^2} \\ \frac{Y_{12}^u}{Y_{12}^d} &\approx \frac{m_s}{m_d} \frac{m_u}{m_c} \quad , \quad \frac{Y_{23}^u}{Y_{23}^d} \approx \frac{m_b}{m_s} \frac{m_c}{m_t} \\ \frac{s_{12}^d}{s_{23}^d} &\approx \frac{s_{12}^u}{s_{23}^u} \approx \frac{Y_{12}^q}{Y_{23}^q} \frac{\mathcal{F}_{23}}{\mathcal{F}_{22}} \quad , \quad \frac{s_{12}^u}{s_{12}^d} \approx \frac{s_{23}^u}{s_{23}^d} \approx \frac{Y_{33}^u}{Y_{33}^d}.\end{aligned}\tag{118}$$

The combination of Eqs. (114)-(118) yields

$$s_{13}^u \approx \left(\frac{Y_{33}^u}{Y_{33}^d}\right)^2 s_{13}^d + \left[\left(\frac{m_c}{m_t}\right)^2 - \left(\frac{Y_{33}^u}{Y_{33}^d}\right)^2 \left(\frac{m_s}{m_b}\right)^2\right] \frac{s_{12}^d}{s_{23}^d}.\tag{119}$$

Imposing now for the sake of simplicity

$$\frac{Y_{33}^u}{Y_{33}^d} = \frac{Y_{23}^u}{Y_{23}^d} \approx \frac{m_b}{m_s} \frac{m_c}{m_t} \equiv r \quad ,\tag{120}$$

one reaches the simplified relationships between mixing angles in the u and d quark sectors:

$$s_{12}^u \approx r s_{12}^d \quad , \quad s_{23}^u \approx r s_{23}^d \quad , \quad s_{13}^u \approx r^2 s_{13}^d \quad (121)$$

Equation (121) allows one to write the $V_{CKM} = (V_L^u)^T V_L^d$ quark mixing matrix in terms of four parameters: r and the three mixing angles s_{12}^d , s_{23}^d , and s_{13}^d . A numerical analysis shows that setting for instance $r = .317239712$, corresponding to using the central values for the quark masses m_s , m_b , m_c , and m_t reported in the Particle Data Group, Ref.[15], and the values $s_{12}^d = 0.32721$, $s_{23}^d = 0.0604208$ and $s_{13}^d = 0.00921978$ yields the quark mixing matrix (ignoring CP violation):

$$V_{CKM} \approx \begin{pmatrix} 0.973776 & 0.227474 & -0.003963 \\ -0.227438 & 0.972890 & -0.041912 \\ -0.005678 & 0.041715 & 0.999113 \end{pmatrix} , \quad (122)$$

Notice that except the matrix element V_{td} , the other eight entries lie within the best fit range values reported in Ref.[15]. These results suggest that the approach given in Eq.(64) for the orthogonal mixing matrices of charged fermions is a good approximation.

6.2 Quark-Lepton complementarity relations

Using the quark mixing angles of Eq.(122) and the range of lepton mixing angles of Eqs. (106)-(111) allows one to obtain the following rough estimation for the quark-lepton complementary relations[16]:

$$\begin{aligned} \theta_{12}^{PMNS} + \theta_{12}^{CKM} &\approx 41.543^\circ - 42.152^\circ \quad , \quad 41.543^\circ - 43.139^\circ \quad , \quad 41.543^\circ - 44.066^\circ , \\ \theta_{23}^{PMNS} + \theta_{23}^{CKM} &\approx 36.835^\circ - 37.184^\circ \quad , \quad 36.835^\circ - 37.754^\circ \quad , \quad 36.835^\circ - 38.295^\circ , \end{aligned} \quad (123)$$

for $s_{13}^2 = 0.034$, 0.037 , and 0.04 , respectively.

7 FCNCs and rare decays for charged leptons

The new exotic scalar particles introduced to implement the radiative mass generation mechanism have the capability to induce FCNCs and contribute to "flavor violation" processes such as $F \rightarrow f_1 f_2 f_3$, to "radiative flavor violating" processes such as $\mu \rightarrow e\gamma$, $\tau \rightarrow \mu\gamma$, $\tau \rightarrow e\gamma$, as well as to the "anomalous magnetic moments" (AMMs) of fermions. In this section I compute roughly these additional contributions for the charged leptons.

Once the generation of fermion masses is completed, the transformations between gauge (0 superscript) and mass (physical) eigenstates are for scalars $\Phi_i = U_{ij}\sigma_j$, Eq.(17), for charged leptons

$$\psi_{eL,eR}^0 = V_{eL,eR} \psi_{eL,eR}, \quad (124)$$

where

$$V_{eL,eR} = V_{eL,eR}^{(1)} V_{eL,eR}^{(2)}, \quad \psi_{eL,eR}^{0T} = (e^0, \mu^0, \tau^0)_{L,R}, \quad \psi_{eL,eR}^T = (e, \mu, \tau)_{L,R}, \quad (125)$$

and analogous transformations for quarks.

7.1 Lepton flavor violation (LFV) processes $F \rightarrow f_1 f_2 f_3$

The scalar fields ($\phi_9, \phi_{10}, \phi_{12}, \phi_{11}$) allow tree level flavor changing vertices through the couplings in Eq.(7). In particular they may induce tree level "lepton flavor violation" (LFV) processes such as $\tau \rightarrow \mu\mu\mu$, $\tau \rightarrow \mu\mu e$, $\tau \rightarrow \mu ee$, $\tau \rightarrow eee$, and $\mu \rightarrow eee$. The generic diagram for these processes is shown in Fig. 4. The decay rate contribution from this generic diagram may be taken as [17]

$$\Gamma(F \rightarrow f_1 f_2 f_3) \approx \frac{m_F^5 Y_l^4}{3072 \pi^3 M_\phi^4}, \quad (126)$$

with Y_l being a coupling constant.

7.1.1 $\mu \rightarrow eee$

Here I discuss some details about the decay $\mu \rightarrow eee$. This rare decay is of particular interest to be analyzed because experimentally it is strongly suppressed. The dominant contribution to this decay comes from the diagrams of Fig. 5. Then, from Eqs.(7) and (126), a rough estimation for this decay rate may be written as

$$\Gamma(\mu \rightarrow eee) \approx \frac{m_\mu^5 Y_{12}^4}{3072 \pi^3} \frac{1}{2} \left\{ (V_{eL})_{21}^2 \left(\sum_k \frac{U_{2k}}{M_k^2} \right)^2 + (V_{eR})_{21}^2 \left(\sum_k \frac{U_{3k}}{M_k^2} \right)^2 \right\}_{\mu eee}, \quad (127)$$

and therefore the branching ratio for this process is³

$$BR(\mu \rightarrow eee) \equiv \frac{\Gamma(\mu \rightarrow eee)}{(\Gamma_\mu)_T} \approx \frac{M_W^4}{g_w^4} Y_{12}^4 \{ \}_{\mu eee} < 1 \times 10^{-12}, \quad (128)$$

where I take

$$(\Gamma_\mu)_T \approx \Gamma(\mu \rightarrow \nu_\mu e \bar{\nu}_e) = \left(\frac{g_W m_\mu}{M_W} \right)^4 \frac{m_\mu}{12(8\pi)^3} \quad (129)$$

³I write the experimental bounds reported in Particle Data Group Ref.[15].

7.1.2 $\tau \rightarrow \mu\mu\mu$

$$\Gamma(\tau \rightarrow \mu\mu\mu) \approx \frac{m_\tau^5 Y_{23}^4}{3072 \pi^3} \frac{1}{2} \left\{ (V_{eL})_{32}^2 \left(\sum_k \frac{U_{1k}}{M_k^2} \right)_L^2 + (V_{eR})_{32}^2 \left(\sum_k \frac{U_{4k}}{M_k^2} \right)_R^2 \right\}_{\tau\mu\mu\mu} \quad (130)$$

Using the mean life of $\tau = (290.6 \pm 1.0) \times 10^{-15}$ s, and hence $(\Gamma_\tau)_T = \frac{1}{\tau} \approx 2.2711631 \times 10^{-12}$ GeV, one gets the branching ratio

$$BR(\tau \rightarrow \mu\mu\mu) \approx C_\tau m_\tau^4 Y_{23}^4 \{ \}_{\tau\mu\mu\mu} < 1.9 \times 10^{-7}, \quad (131)$$

with $C_\tau \equiv 4.107105 \times 10^6$

7.1.3 $\tau^- \rightarrow \mu^+ \mu^- e^-$

$$\Gamma(\tau^- \rightarrow \mu^+ \mu^- e^-) \approx \frac{m_\tau^5 (Y_{12} Y_{23})^2}{3072 \pi^3} \frac{1}{2} \left\{ \left(\sum_k \frac{U_{1k} U_{2k}}{M_k^2} \right)^2 + \left(\sum_k \frac{U_{3k} U_{4k}}{M_k^2} \right)^2 \right\}_{\tau\mu\mu e}, \quad (132)$$

with the branching ratio

$$BR(\tau^- \rightarrow \mu^+ \mu^- e^-) \approx C_\tau m_\tau^4 (Y_{12} Y_{23})^2 \{ \}_{\tau\mu\mu e} < 2 \times 10^{-7} \quad (133)$$

7.1.4 $\tau^- \rightarrow \mu^+ e^- e^-$

$$\Gamma(\tau^- \rightarrow \mu^+ e^- e^-) \approx \frac{m_\tau^5 (Y_{12} Y_{23})^2}{3072 \pi^3} \frac{1}{2} \left\{ (V_{eL})_{21}^2 \left(\sum_k \frac{U_{1k} U_{2k}}{M_k^2} \right)^2 + (V_{eR})_{21}^2 \left(\sum_k \frac{U_{3k} U_{4k}}{M_k^2} \right)^2 \right\}_{\tau\mu ee}, \quad (134)$$

$$BR(\tau^- \rightarrow \mu^+ e^- e^-) \approx C_\tau m_\tau^4 (Y_{12} Y_{23})^2 \{ \}_{\tau\mu ee} < 1.1 \times 10^{-7} \quad (135)$$

7.1.5 $\tau \rightarrow eee$

$$\Gamma(\tau \rightarrow eee) \approx \frac{m_\tau^5 (Y_{12} Y_{23})^2}{3072 \pi^3} \frac{1}{2} \left\{ (V_{eL})_{21}^4 \left(\sum_k \frac{U_{1k} U_{2k}}{M_k^2} \right)^2 + (V_{eR})_{21}^4 \left(\sum_k \frac{U_{3k} U_{4k}}{M_k^2} \right)^2 \right\}_{\tau eee}, \quad (136)$$

$$BR(\tau \rightarrow eee) \approx C_\tau m_\tau^4 (Y_{12} Y_{23})^2 \{ \}_{\tau eee} < 2 \times 10^{-7}. \quad (137)$$

7.2 Anomalous magnetic moments and radiative rare decays $F \rightarrow f\gamma$

The amplitude for the radiative process $f_1 \rightarrow f_2 \gamma$ with f_1 and f_2 being two equally charged fermions and γ a real photon is written as [18]

$$i\mathcal{M}(f_1(p_1) \rightarrow f_2(p_2) + \gamma) = i\bar{u}_2(p_2) \left(\epsilon^\mu \gamma_\mu F_1^V(0) \delta_{f_1 f_2} + \frac{\sigma_{\mu\nu} q^\nu \epsilon^\mu}{m_1 + m_2} (F_2^V(0) + F_2^A(0) \gamma_5) \right) u_1(p_1), \quad (138)$$

where $F_2^{V(A)}$ gives the AMM (electric dipole moment) for the fermion f_1 when $f_1 = f_2$. The generic diagrams for the process $f_{1L} \rightarrow f_{2R} \gamma$ are shown in Fig. 6, in these diagrams σ stands for a mass eigenstate scalar field. The respective evaluation of these diagrams gives

$$iA_L \approx \frac{Y_l^2 q_e}{16\pi^2} N(M_k, m_i) \bar{e}_{2R} i\sigma^{\mu\nu} q_\nu \epsilon_\mu e_{1L} \quad \text{and} \quad iA_R \approx \frac{Y_l^2 q_e}{16\pi^2} N(M_k, m_i) \bar{e}_{2L} i\sigma^{\mu\nu} q_\nu \epsilon_\mu e_{1R}, \quad (139)$$

where the second amplitude comes from the diagrams where L and R are interchanged, and $N(M_k, m_i)$ may be approximated as

$$N(M_k, m_i) \approx \frac{m_i}{M_k^2} \ln \frac{M_k^2}{m_i^2} \quad (140)$$

in the limit $M_k \gg m_i$. Notice that due to scalar field mixing the contribution of these loops is finite as those in the mass case.

Because of the fermion mixing matrices structure the diagrams that make the largest contribution to the AMMs of the charged leptons are, for the electron, the diagram with the muon inside the loop, and for the muon and tau, the diagrams with tau as the internal fermion.

7.2.1 Muon anomalous magnetic moment

The dominant contribution for the muon AMM comes from the diagram of Fig. 7, where the insertion of a photon on the internal lines is understood as in the generic diagrams of Fig. 6. The expression for this scalar contribution is [18, 19]

$$\begin{aligned} a_\mu &= \frac{m_\mu Y_{23}^2}{16\pi^2} (V_{eL})_{22} (V_{eR})_{22} (G^{\mu L} + G^{\mu R}) \approx \frac{m_\mu Y_{23}^2}{16\pi^2} (G^{\mu L} + G^{\mu R}) \\ &\approx \frac{m_\mu m_\tau Y_{23}^2}{8\pi^2} \sum_k \frac{U_{1k} U_{4k}}{M_k^2} \ln \frac{M_k^2}{m_\tau^2} \end{aligned} \quad (141)$$

where

$$\begin{aligned} G^{\mu L} = G^{\mu R} &= \sum_{k,i} U_{1k} U_{4k} (V_{eL})_{3i} (V_{eR})_{3i} N(M_k, m_i) \approx \sum_k U_{1k} U_{4k} N(M_k, m_\tau) \\ &\approx m_\tau \sum_k \frac{U_{1k} U_{4k}}{M_k^2} \ln \frac{M_k^2}{m_\tau^2}. \end{aligned} \quad (142)$$

7.2.2 Electron and tau anomalous magnetic moments

Performing a similar analysis for e and τ leptons, one gets

For electron:

$$a_e \approx \frac{m_e Y_{12}^2}{16\pi^2} (G^{eL} + G^{eR}) \approx \frac{m_e m_\mu Y_{12}^2}{8\pi^2} \sum_k \frac{U_{2k} U_{3k}}{M_k^2} \ln \frac{M_k^2}{m_\mu^2} \quad (143)$$

where

$$\begin{aligned}
G^{eL} = G^{eR} &= \sum_{k,i} U_{2k} U_{3k} (V_{eL})_{2i} (V_{eR})_{2i} N(M_k, m_i) \approx \sum_k U_{2k} U_{3k} N(M_k, m_\mu) \\
&\approx m_\mu \sum_k \frac{U_{2k} U_{3k}}{M_k^2} \ln \frac{M_k^2}{m_\mu^2} .
\end{aligned} \tag{144}$$

For tau:

$$a_\tau \approx \frac{m_\tau Y_{33}^2}{16\pi^2} (G^{\tau L} + G^{\tau R}) \approx \frac{m_\tau^2 Y_{33}^2}{8\pi^2} \sum_k \frac{U_{2k} U_{3k}}{M_k^2} \ln \frac{M_k^2}{m_\tau^2} \tag{145}$$

where

$$\begin{aligned}
G^{\tau L} = G^{\tau R} &= \sum_{k,i} U_{2k} U_{3k} (V_{eL})_{3i} (V_{eR})_{3i} N(M_k, m_i) \approx \sum_k U_{2k} U_{3k} N(M_k, m_\tau) \\
&\approx m_\tau \sum_k \frac{U_{2k} U_{3k}}{M_k^2} \ln \frac{M_k^2}{m_\tau^2} .
\end{aligned} \tag{146}$$

7.2.3 Radiative decay $\mu \rightarrow e\gamma$

A similar analysis to the one for the muon AMM leads to the decay rate

$$\begin{aligned}
\Gamma(\mu \rightarrow e\gamma) &= (m_\mu + m_e)^2 \frac{m_\mu (Y_{12} Y_{23})^2}{(16)^3 \pi^5} \left(1 - \frac{m_e}{m_\mu}\right)^2 \left(1 - \frac{m_e^2}{m_\mu^2}\right) (|(V_{eL})_{22} (V_{eR})_{11} G^{\mu L e R}|^2 + |(V_{eL})_{11} (V_{eR})_{22} G^{\mu R e L}|^2) \\
&\approx \frac{m_\mu^3 (Y_{12} Y_{23})^2}{(16)^3 \pi^5} (|G^{\mu L e R}|^2 + |G^{\mu R e L}|^2) \\
&\approx \frac{m_\mu^3 m_\tau^2 (Y_{12} Y_{23})^2}{(16)^3 \pi^5} \left\{ (V_{eR})_{23}^2 \left(\sum_k \frac{U_{1k} U_{3k}}{M_k^2} \ln \frac{M_k^2}{m_\tau^2} \right)^2 + (V_{eL})_{23}^2 \left(\sum_k \frac{U_{2k} U_{4k}}{M_k^2} \ln \frac{M_k^2}{m_\tau^2} \right)^2 \right\}_{\mu e \gamma} ,
\end{aligned} \tag{147}$$

$$\begin{aligned}
G^{\mu L e R} &= \sum_{k,i} U_{1k} U_{3k} (V_{eL})_{3i} (V_{eR})_{2i} N(M_k, m_i) \approx (V_{eR})_{23} m_\tau \sum_k \frac{U_{1k} U_{3k}}{M_k^2} \ln \frac{M_k^2}{m_\tau^2} , \\
G^{\mu R e L} &= \sum_{k,i} U_{2k} U_{4k} (V_{eL})_{2i} (V_{eR})_{3i} N(M_k, m_i) \approx (V_{eL})_{23} m_\tau \sum_k \frac{U_{2k} U_{4k}}{M_k^2} \ln \frac{M_k^2}{m_\tau^2} .
\end{aligned} \tag{148}$$

The resulting branching ratio may be expressed as

$$BR(\mu \rightarrow e\gamma) \approx \frac{3}{2\pi^2} \frac{m_\tau^2}{m_\mu^2} \left(\frac{M_W}{g_w} \right)^4 (Y_{12} Y_{23})^2 \{ \}_{\mu e \gamma} < 1.2 \times 10^{-11} . \tag{149}$$

7.2.4 Radiative decays $\tau \rightarrow \mu\gamma$ and $\tau \rightarrow e\gamma$

Carrying out a similar analysis, one gets

$\tau \rightarrow \mu\gamma$:

$$\begin{aligned}
\Gamma(\tau \rightarrow \mu\gamma) &\approx \frac{m_\tau^3 (Y_{23} Y_{33})^2}{(16)^3 \pi^5} (|G^{\tau L \mu R}|^2 + |G^{\tau R \mu L}|^2) \\
&\approx \frac{m_\tau^5 (Y_{23} Y_{33})^2}{(16)^3 \pi^5} \left\{ \left(\sum_k \frac{U_{2k} U_{4k}}{M_k^2} \ln \frac{M_k^2}{m_\tau^2} \right)^2 + \left(\sum_k \frac{U_{1k} U_{3k}}{M_k^2} \ln \frac{M_k^2}{m_\tau^2} \right)^2 \right\}_{\tau\mu\gamma}
\end{aligned} \tag{150}$$

where

$$\begin{aligned}
G^{\tau L \mu R} &= \sum_{k,i} U_{2k} U_{4k} (V_{eL})_{3i} (V_{eR})_{3i} N(M_k, m_i) \approx m_\tau \sum_k \frac{U_{2k} U_{4k}}{M_k^2} \ln \frac{M_k^2}{m_\tau^2}, \\
G^{\tau R \mu L} &= \sum_{k,i} U_{1k} U_{3k} (V_{eL})_{3i} (V_{eR})_{3i} N(M_k, m_i) \approx m_\tau \sum_k \frac{U_{1k} U_{3k}}{M_k^2} \ln \frac{M_k^2}{m_\tau^2},
\end{aligned} \tag{151}$$

and branching ratio

$$BR(\tau \rightarrow \mu\gamma) \approx C'_\tau m_\tau^4 (Y_{23} Y_{33})^2 \{ \}_{\tau\mu\gamma} < 6.8 \times 10^{-8}, \tag{152}$$

with $C'_\tau = 6.24205152 \times 10^5$.

$\tau \rightarrow e\gamma$:

$$\begin{aligned}
\Gamma(\tau \rightarrow e\gamma) &\approx \frac{m_\tau^3 (Y_{12} Y_{23})^2}{(16)^3 \pi^5} (|G^{\tau L e R}|^2 + |G^{\tau R e L}|^2) \\
&\approx \frac{m_\tau^3 m_\mu^2 (Y_{12} Y_{23})^2}{(16)^3 \pi^5} \left\{ \left(\sum_k \frac{U_{1k} U_{3k}}{M_k^2} \ln \frac{M_k^2}{m_\mu^2} \right)^2 + \left(\sum_k \frac{U_{2k} U_{4k}}{M_k^2} \ln \frac{M_k^2}{m_\mu^2} \right)^2 \right\}_{\tau e\gamma},
\end{aligned} \tag{153}$$

where

$$\begin{aligned}
G^{\tau L e R} &= \sum_{k,i} U_{1k} U_{3k} (V_{eL})_{2i} (V_{eR})_{2i} N(M_k, m_i) \approx m_\mu \sum_k \frac{U_{1k} U_{3k}}{M_k^2} \ln \frac{M_k^2}{m_\mu^2}, \\
G^{\tau R e L} &= \sum_{k,i} U_{2k} U_{4k} (V_{eL})_{2i} (V_{eR})_{2i} N(M_k, m_i) \approx m_\mu \sum_k \frac{U_{2k} U_{4k}}{M_k^2} \ln \frac{M_k^2}{m_\mu^2},
\end{aligned} \tag{154}$$

and branching ratio

$$BR(\tau \rightarrow e\gamma) \approx C'_\tau m_\tau^2 m_\mu^2 (Y_{12} Y_{23})^2 \{ \}_{\tau e\gamma} < 1.1 \times 10^{-7}. \tag{155}$$

8 Summary and conclusions

I have reported a detailed analysis on fermion masses and mixing, including neutrino mixing, within the context of an extension of the standard model with an $U(1)_H$ flavor symmetry and hierarchical radiative mass mechanism [2]. The results of this analysis show that this model has the capability to accommodate the observed spectrum of quark masses and mixing angles in the V_{CKM} , as it is shown through the analysis in Secs. 3 and 6. In a similar way the spectrum of charged lepton masses is consistently generated through the analysis presented in Secs. 3 and 4. Upper bounds for the charged lepton mixing angles are given in Eq. (83). These upper bounds imply that mixing in the lepton sector comes almost completely from neutrino mixing; that is, $V_{PMNS} \approx V_\nu$. In this approach all lepton mixing elements in V_{PMNS} are written completely in terms of neutrino masses. A numerical analysis shows that using $0.033 \lesssim s_{13}^2 \lesssim 0.04$ one gets large mixing angles for θ_{12} and θ_{23} , $0.7 \leq \sin^2 2\theta_{12} \leq 0.777209$ and $0.87 \leq \sin^2 2\theta_{23} \leq 0.902305$, within the present allowed 3σ limits as reported by recent global analysis of neutrino data oscillation [8, 13, 14]. Using these allowed ranges of values for quark and lepton mixing, predictions for neutrino masses and quark-lepton complementary relations are given in the Eqs. (112), (113), and (123), respectively.

From the phenomenological point of view it is interesting to look for a set of scalar mass parameters in M_ϕ^2 , Eq. (165), that allows us to account for the strong experimental suppression on LFV processes, such as $\mu \rightarrow eee$, radiative rare decays $\mu \rightarrow e\gamma$, $\tau \rightarrow \mu\gamma$, $\tau \rightarrow e\gamma$, and the muon anomalous magnetic moment. To achieve this goal a detailed numerical analysis and fit it is needed, trying to keep at least the lowest scalar mass eigenvalue η_1 within few TeV^2 . However, it is important to comment that Eq. (83) gives a good approximation for the upper bounds on charged lepton mixing angles, and hence $V_{PMNS} \approx V_\nu$ would remain as a good approach in this model.

Thus, the contribution of my analysis in comparison to the one realized in Ref. [2] may be summarized in the following aspects:

- *Scalar sector:* I have performed the analysis by considering the most general structure for M_ϕ^2 .
- *Charged fermion sector:*
 - I have obtained and then diagonalized the quark and lepton mass matrices at one and two loops in close analytical form.
 - Taking advantage of the strong hierarchy of quark and charged lepton masses, approximate expressions for the orthogonal mixing matrices of charged fermions are obtained.
 - I have reported general analytical expressions for the branching ratios of LFV processes, radiative rare decays and for the AMMs of charged leptons.
- *Neutrinos:* The V_{PMNS} lepton mixing matrix is obtained and written completely in terms of the neutrino masses, and numerical results for lepton mixing angles are provided.

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9 Appendix: Diagonalization of a generic real symmetric 3x3 mass matrix

In this appendix I give the details to diagonalize a generic real symmetric mass matrix defined as

$$M \equiv \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}. \quad (156)$$

One can diagonalize this matrix M through the orthogonal matrix V as $V^T M V = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$, λ_i , $i = 1, 2, 3$ being the eigenvalues of M . The determinant equation $\det[M - \lambda] = 0$ imposes the constraint that each one of the eigenvalues λ_i obeys the cubic equation

$$-\lambda^3 + (a + b + c)\lambda^2 - (ab - d^2 + ac - e^2 + bc - f^2)\lambda + abc - f^2a - e^2b - d^2c + 2def = 0. \quad (157)$$

Thus, from Eq. (157) one obtains the following nonlinear relationships to hold:

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &= a + b + c \\ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 &= ab - d^2 + ac - e^2 + bc - f^2 \\ \lambda_1\lambda_2\lambda_3 &= abc - f^2a - e^2b - d^2c + 2def. \end{aligned} \quad (158)$$

I do not impose any hierarchy between the eigenvalues λ_1 , λ_2 , and λ_3 . However, I assume they are nondegenerated. Computing now the eigenvectors⁴, the orthogonal matrix V may be writing as

$$V = \begin{pmatrix} x & y \frac{F_1(\lambda_2)}{\Delta_2(\lambda_2)} & z \frac{F_2(\lambda_3)}{\Delta_3(\lambda_3)} \\ x \frac{F_1(\lambda_1)}{\Delta_1(\lambda_1)} & y & z \frac{F_3(\lambda_3)}{\Delta_3(\lambda_3)} \\ x \frac{F_2(\lambda_1)}{\Delta_1(\lambda_1)} & y \frac{F_3(\lambda_2)}{\Delta_2(\lambda_2)} & z \end{pmatrix}, \quad (159)$$

where x , y , and z are normalization constants, and the functions involved are defined as

$$\begin{aligned} \Delta_1(\lambda) &\equiv (b - \lambda)(c - \lambda) - f^2, & F_1(\lambda) &\equiv -d(c - \lambda) + ef, \\ \Delta_2(\lambda) &\equiv (a - \lambda)(c - \lambda) - e^2, & F_2(\lambda) &\equiv -e(b - \lambda) + df, \\ \Delta_3(\lambda) &\equiv (a - \lambda)(b - \lambda) - d^2, & F_3(\lambda) &\equiv -f(a - \lambda) + de. \end{aligned} \quad (160)$$

Using properly Eqs. (157) and (158), it is possible to check the orthogonality between columns(eigenvectors) of V . Moreover, the functions $\Delta_i(\lambda)$ and $F_i(\lambda)$ in Eq.(160) satisfy the important and useful relationships

⁴Here still unnormalized

$$\begin{aligned}
F_1^2(\lambda) &= \Delta_1(\lambda)\Delta_2(\lambda) \quad , \quad F_1(\lambda)F_2(\lambda) = \Delta_1(\lambda)F_3(\lambda) \, , \\
F_2^2(\lambda) &= \Delta_1(\lambda)\Delta_3(\lambda) \quad , \quad F_1(\lambda)F_3(\lambda) = \Delta_2(\lambda)F_2(\lambda) \, , \\
F_3^2(\lambda) &= \Delta_2(\lambda)\Delta_3(\lambda) \quad , \quad F_2(\lambda)F_3(\lambda) = \Delta_3(\lambda)F_1(\lambda) \, .
\end{aligned} \tag{161}$$

Defining

$$\begin{aligned}
h(\lambda) \equiv \Delta_1(\lambda) + \Delta_2(\lambda) + \Delta_3(\lambda) &= 3\lambda^2 - 2(a+b+c)\lambda + ab - d^2 + ac - e^2 + bc - f^2 \\
&= 3\lambda^2 - 2(\lambda_1 + \lambda_2 + \lambda_3)\lambda + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \, ,
\end{aligned} \tag{162}$$

explicitly,

$$\begin{aligned}
h(\lambda_1) &= \lambda_1^2 - (\lambda_2 + \lambda_3)\lambda_1 + \lambda_2\lambda_3 = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1) \, , \\
h(\lambda_2) &= \lambda_2^2 - (\lambda_1 + \lambda_3)\lambda_2 + \lambda_1\lambda_3 = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2) \, , \\
h(\lambda_3) &= \lambda_3^2 - (\lambda_1 + \lambda_2)\lambda_3 + \lambda_1\lambda_2 = (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3) \, .
\end{aligned} \tag{163}$$

One can use now the relationships of Eqs. (161) and (163) to normalize the eigenvectors, obtaining that in general the square matrix elements V_{ij}^2 $i, j = 1, 2, 3$ may be expressed as

$$V_{ij}^2 = \frac{\Delta_i(\lambda_j)}{h(\lambda_j)} \geq 0 \quad \text{and hence} \quad |V_{ij}| = \sqrt{\frac{\Delta_i(\lambda_j)}{h(\lambda_j)}}. \tag{164}$$

Equation (164) defines the magnitudes for the matrix elements V_{ij} in Eq.(159). Setting now the diagonal elements of V as positives, $x > 0, y > 0$, and $z > 0$, the signs of the off diagonal elements, V_{ij} , $i \neq j$, may be obtained directly from the Eq.(159) in a particular set of giving parameters a, b, c, d, e , and f that define the real symmetric mass matrix M in Eq. (156).

It is important to mention here that the method introduced in this Appendix to diagonalize a generic 3x3 real symmetric mass matrix agrees with the diagonalization performed in Ref. [20] for the special case of Fritzsch's ansatz, $a = e = 0$.

9.1 Diagonalization of the generic exotic scalar mass matrices

The most general square scalar mass matrix for the exotic scalar fields, which mediate the radiative mass generation of the light fermions at one and two loops in the u, d, and e charged fermion sectors, may be written as

$$M_\phi^2 = \begin{pmatrix} a_1 & b & 0 & 0 \\ b & a_2 & c & 0 \\ 0 & c & a_3 & d \\ 0 & 0 & d & a_4 \end{pmatrix}. \tag{165}$$

This matrix may be diagonalized through the orthogonal matrix U as $U^T M_\phi^2 U = \text{diag}(\eta_1, \eta_2, \eta_3, \eta_4)$, $\eta_i \equiv M_i^2$, $i = 1, 2, 3, 4$ being the eigenvalues of M_ϕ^2 . Using the same procedure and method introduced previously, the orthogonal matrix U may be writing as

$$U = \begin{pmatrix} x' & y' \frac{f_2(\eta_2)}{\Delta_2(\eta_2)} & z' \frac{f_3(\eta_3)}{\Delta_3(\eta_3)} & t' \frac{f_4(\eta_4)}{\Delta_4(\eta_4)} \\ x' \frac{f_2(\eta_1)}{\Delta_1(\eta_1)} & y' & z' \frac{g_3(\eta_3)}{\Delta_3(\eta_3)} & t' \frac{g_4(\eta_4)}{\Delta_4(\eta_4)} \\ x' \frac{f_3(\eta_1)}{\Delta_1(\eta_1)} & y' \frac{g_3(\eta_2)}{\Delta_2(\eta_2)} & z' & t' \frac{h_4(\eta_4)}{\Delta_4(\eta_4)} \\ x' \frac{f_4(\eta_1)}{\Delta_1(\eta_1)} & y' \frac{g_4(\eta_2)}{\Delta_2(\eta_2)} & z' \frac{h_4(\eta_3)}{\Delta_3(\eta_3)} & t' \end{pmatrix}, \quad (166)$$

where x' , y' , z' , and t' are normalization constants, and the functions involved are defined as

$$\begin{aligned} \Delta_1(\eta) &\equiv (a_2 - \eta)(a_3 - \eta)(a_4 - \eta) - (a_2 - \eta)d^2 - (a_4 - \eta)c^2, \\ \Delta_2(\eta) &\equiv (a_1 - \eta) [(a_3 - \eta)(a_4 - \eta) - d^2], \\ \Delta_3(\eta) &\equiv (a_4 - \eta) [(a_1 - \eta)(a_2 - \eta) - b^2], \\ \Delta_4(\eta) &\equiv (a_1 - \eta)(a_2 - \eta)(a_3 - \eta) - (a_1 - \eta)c^2 - (a_3 - \eta)b^2, \end{aligned} \quad (167)$$

and

$$\begin{aligned} f_2(\eta) &\equiv -b [(a_3 - \eta)(a_4 - \eta) - d^2] & , & \quad g_3(\eta) \equiv -c(a_1 - \eta)(a_4 - \eta), \\ f_3(\eta) &\equiv bc(a_4 - \eta) & , & \quad g_4(\eta) \equiv cd(a_1 - \eta), \\ f_4(\eta) &\equiv -bcd & , & \quad h_4(\eta) \equiv -d [(a_1 - \eta)(a_2 - \eta) - b^2]. \end{aligned} \quad (168)$$

These functions satisfy relationships analogous to those of Eq.(161), allowing us to obtain the normalization constants and then to write the square matrix elements as

$$U_{ij}^2 = \frac{\Delta_i(\eta_j)}{h(\eta_j)} \geq 0 \quad , \quad \text{and hence} \quad |U_{ij}| = \sqrt{\frac{\Delta_i(\eta_j)}{h(\eta_j)}}, \quad (169)$$

where

$$\begin{aligned} h(\eta_1) &= (\eta_2 - \eta_1)(\eta_3 - \eta_1)(\eta_4 - \eta_1) & , & \quad h(\eta_2) = (\eta_1 - \eta_2)(\eta_3 - \eta_2)(\eta_4 - \eta_2), \\ h(\eta_3) &= (\eta_1 - \eta_3)(\eta_2 - \eta_3)(\eta_4 - \eta_3) & , & \quad h(\eta_4) = (\eta_1 - \eta_4)(\eta_2 - \eta_4)(\eta_3 - \eta_4), \end{aligned} \quad (170)$$

and leads to the useful equalities

$$U_{1k}U_{4k} = \frac{f_4(\eta_k)}{h(\eta_k)} \quad , \quad U_{2k}U_{4k} = \frac{g_4(\eta_k)}{h(\eta_k)} \quad , \quad U_{1k}U_{3k} = \frac{f_3(\eta_k)}{h(\eta_k)} \quad , \quad U_{2k}U_{3k} = \frac{g_3(\eta_k)}{h(\eta_k)}. \quad (171)$$

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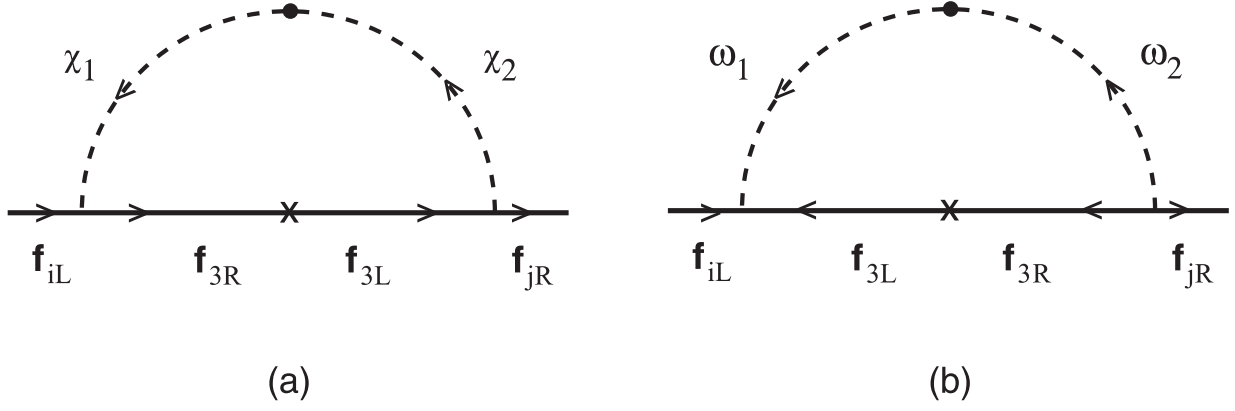


Figure 1: Generic diagrams contributing to fermion masses. (a) Dirac-type couplings, (b) Majorana-type couplings.

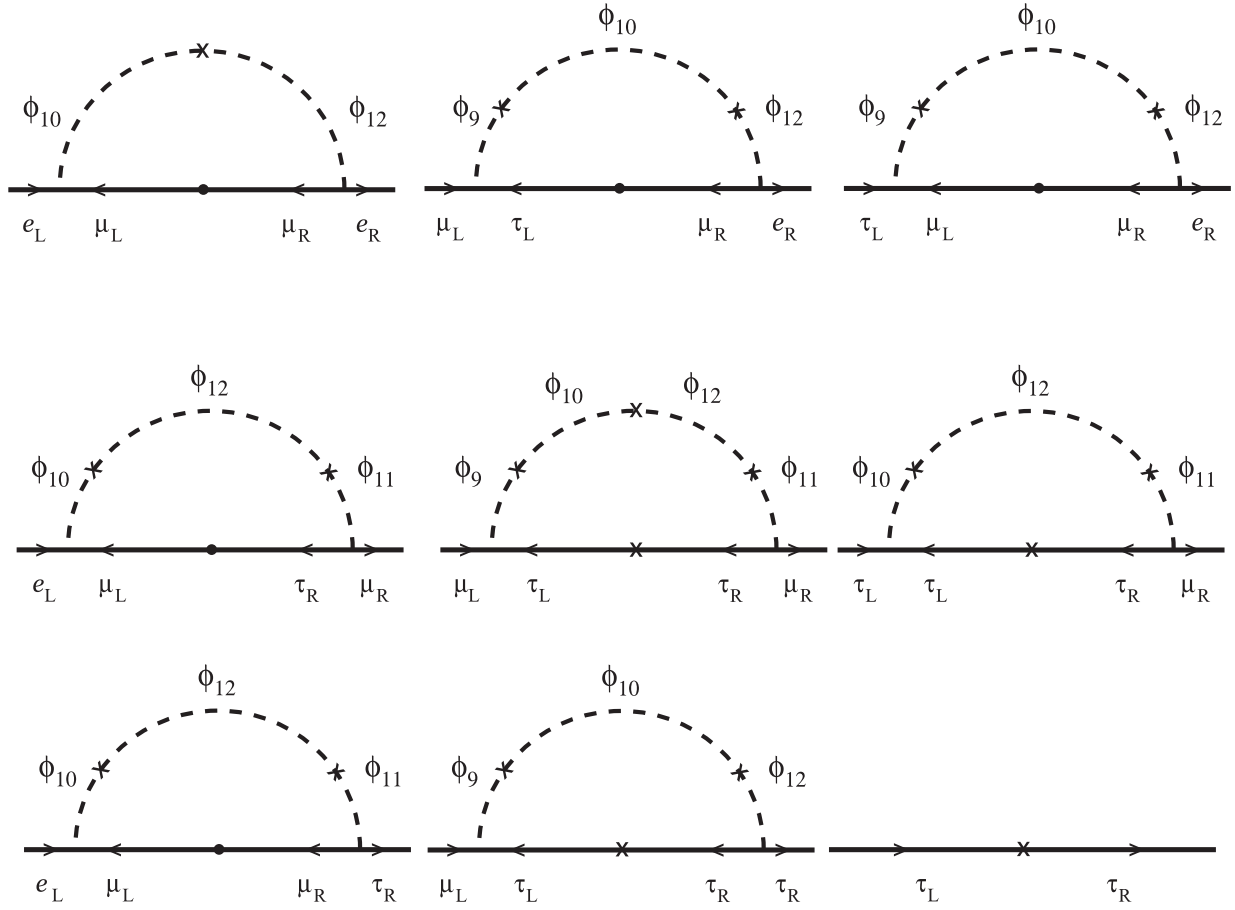


Figure 2: Mass diagrams for the charged lepton sector.

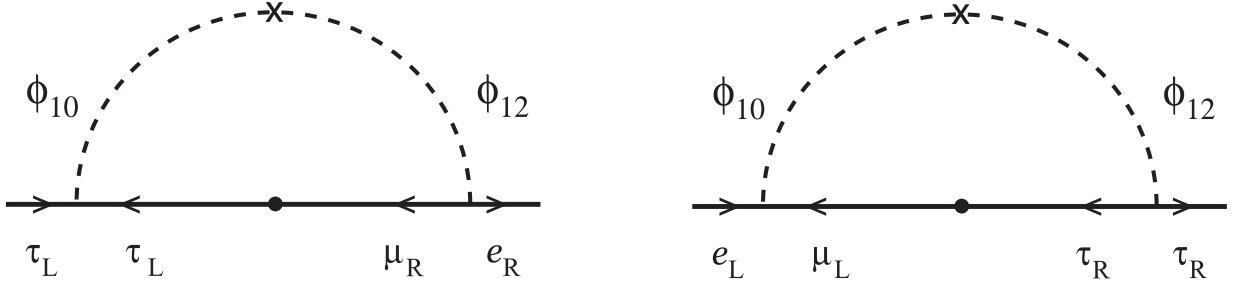


Figure 3: Additional diagrams for the entries (1,3) and (3,1).

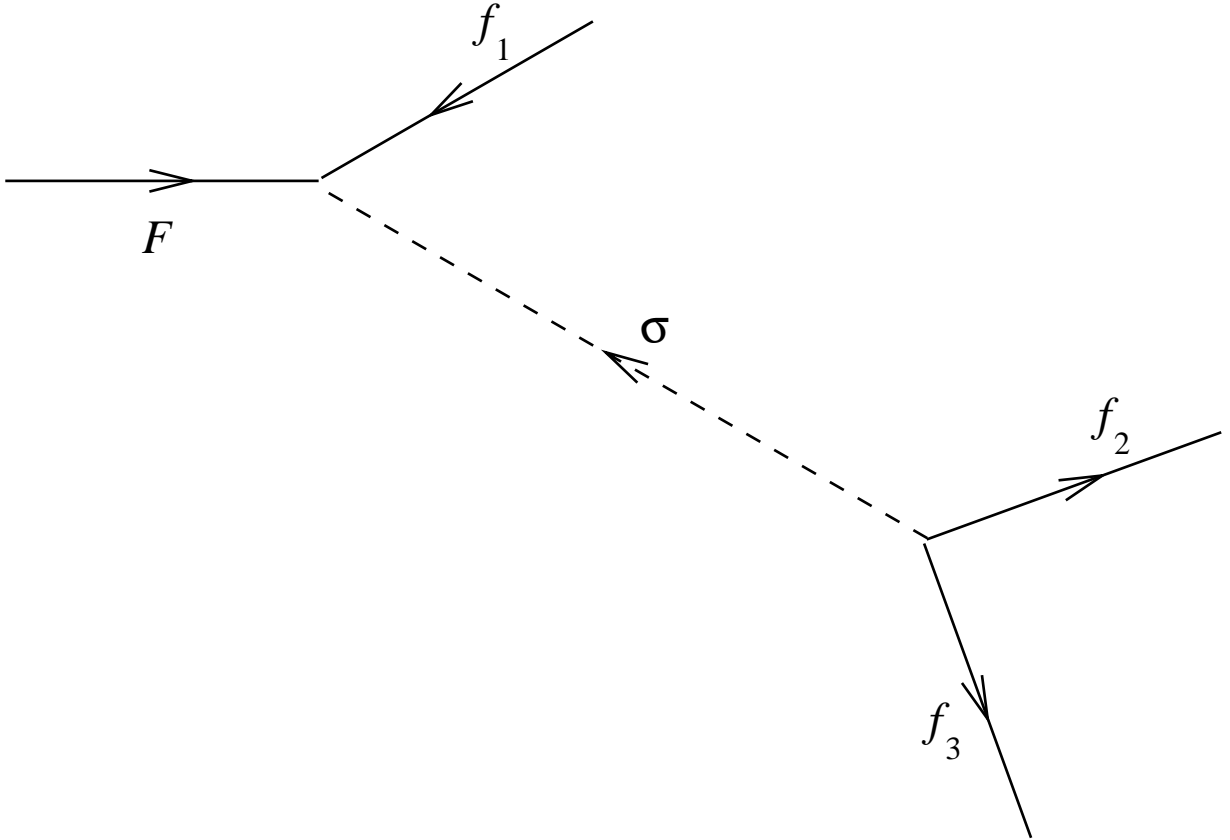


Figure 4: Generic diagram for the LFV processes $F \rightarrow f_1 f_2 f_3$.

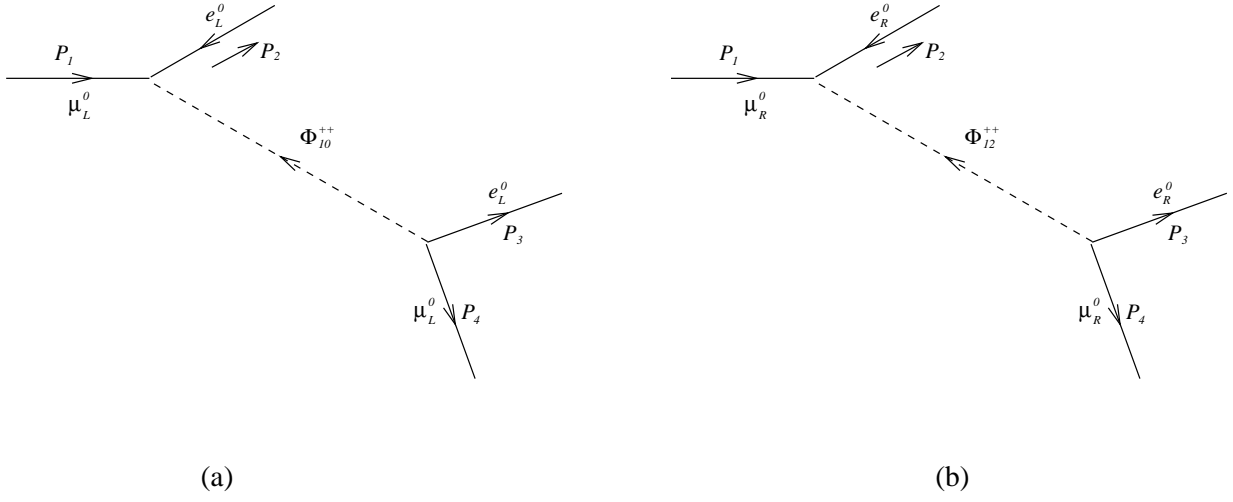


Figure 5: Diagrams for the rare decay $\mu \rightarrow eee$.

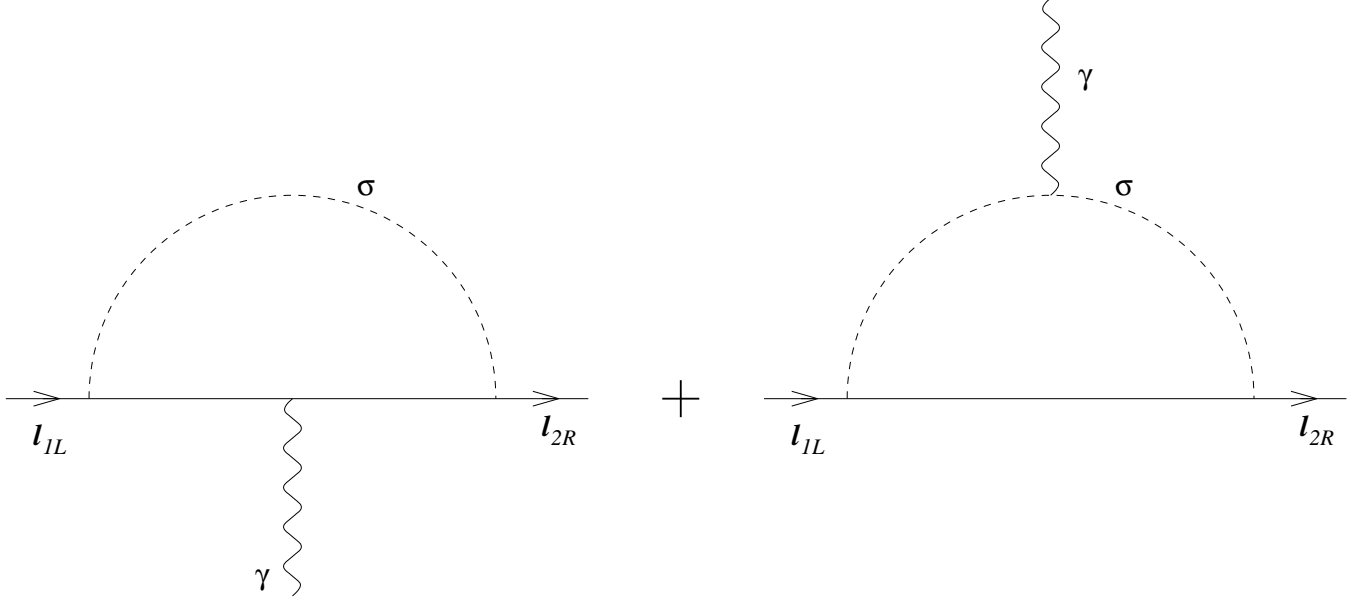


Figure 6: Generic diagrams for the process $l_1 \rightarrow l_2 \gamma$, where a scalar mass eigenstate σ is involved.

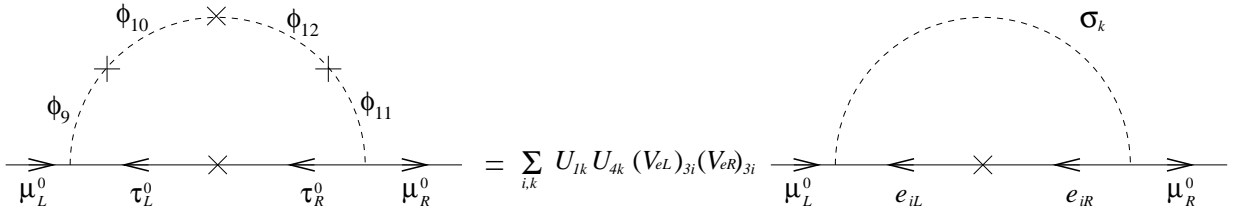


Figure 7: Main contribution to the muon anomalous magnetic moment (the insertion of a photon on the internal lines is understood as in the Fig. 6).